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CONSTRUCTION OF PLANE CURVES OF GIVEN ORDER AND GENUS, HAVING DISTINCT DOUBLE POINTS.

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IN researches in the theory of birational transformations it is frequently desirable to employ curves of given order and given genus, all of whose singularities are ordinary distinct double points; but the possibility of finding such curves has been assumed. In the following note I show that such curves exist for every value of the genus p not exceeding $\frac{1}{2}(n-1)(n-2)$, n being the order of the curve, and determine the equation in each case.

1. Points on the quadric surface $F \equiv xz - yw = 0$ may be defined by simultaneous values of $x_1 : x_2$ and $y_1 : y_2$, where

$$\frac{x}{y} = \frac{w}{z} = \frac{x_1}{x_2}, \quad \frac{x}{w} = \frac{y}{z} = \frac{y_1}{y_2}.$$

An algebraic curve lying on F_2, cutting the generators of one system in r points, and those of the other in $n - r$ points may be expressed by an equation of the form

$$f\left(\overset{r}{\frac{x_1}{x_2}}, \overset{n-r}{\frac{y_1}{y_2}}\right) = 0 \qquad (r \leqq n - r).$$

By multiplying this equation by a suitable power of x_1 and making use of the relations

$$x = x_1 y_2, \quad y = x_2 y_1, \quad z = x_2 y_2, \quad w = x_1 y_1,$$

the equation $f=0$ may be transformed into that of a surface F_{n-r} containing our space curve R_n, and whose residual intersection with F_2 consists of a generator contained $(n-2r)$-fold. If now between $F_2=0$, $F_{n-r}=0$ the variable y be eliminated, the resulting equation $K_n(x, z, w)=0$ is that of a cone with vertex at $(0, 1, 0, 0)$ on F_2. The section made on this cone by the plane $y=0$ is a curve C_n having an r-fold point at $(0, 0, 1, 0)$ and an $(n-r)$-fold point at $(1, 0, 0, 0)$, and no other singularities, when the coefficients of $f=0$ are so chosen that the correspondence has no double coincidences.

If y be now transformed into $y-w$, the other variables remaining fixed, the equation of the cone will not be changed, but F_2 becomes $w^2+yw-xz=0$. The relations between the coordinates (x, y, z, w) of a point on F_2 and its projection (ξ, ζ, ω) in $y=0$ are found to be

$$\sigma x=\xi\omega, \quad \sigma y=\xi\zeta-\omega^2, \quad \sigma z=\zeta\omega, \quad \sigma w=\omega^2.$$

It is an ordinary stereographic projection, and therefore birational.

Again, if between $F_{n-r}=0$ and $F_2=0$ the variable w be eliminated, the resulting equation $K'_n(x, y, z)=0$ will define a cone from $(0, 0, 0, 1)$ as vertex and containing our curve. Since the vertex is not on F_2 and since no line other than the generators can cut F_2 in more than two points, a plane section of K'_n can contain no higher singularities than double points.

This idea can now be applied to the transformation of plane curves, without any reference to the configuration in space. By superposing the two planes (ξ, ζ, ω), (ξ', η', ζ') in such a way that $\eta'=\omega$, the whole process can be carried on in the plane. The result can be stated as follows:

Given the plane curve $f_n(x, y, z)=0$ *having an r-fold point at* $(1, 0, 0)$ *and an* $(n-r)$*-fold point at* $(1, 0, 0)$ *but no other point singularities. It can be birationally transformed into another curve of the same order and having only distinct double points by means of the transformation*

$$\rho x'=xz, \quad \rho y'=xy-z^2, \quad \rho z'=yz.$$

In particular, if $r=1$, f is rational; hence *rational curves of every order exist, having distinct double points.* Thus, in non-homogeneous form, the general parabola $y=xf_{n-1}(x)$ can be transformed into another rational curve of the same order by eliminating x, y from the equations

$$y = xf_{n-1}(x), \quad x'y = x, \quad y y = xy - 1,$$

the new curve in x', y' having distinct double points. For $r = 2$, hyperelliptic curves of genus $n - 3$, i. e., having

$$h = \tfrac{1}{2}(n - 2)(n - 3) + 1$$

double points, result. To obtain similar curves of lower genera, it is necessary to impose double coincidences in the correspondence $f(x_1 : x_2, y_1 : y_2) = 0$. A more general method for obtaining curves which are not hyperelliptic and whose genus always lies in the same interval can be at once derived from the theorem of Halphen * that every plane curve of order n having at least h double points can be obtained by projecting some non-singular space curve on the plane. It is only necessary that the center of perspective be not on the ruled surface of trisecants of the space curve.

The method of Halphen † can also be applied in case $h \geqq \frac{1}{3}(n - 1)(n - 2)$, but for lower values down to the largest integer contained in $[\frac{1}{2}(n - 1)]^2$ this is no longer possible. But for this interval the number of conditions to be satisfied is sufficiently small to insure that the required number of actual double points can be imposed on the space curve without making it composite. The number of double points is less than $\frac{1}{12}(n - 5)(n - 1)$, while the number of constants still at our disposal is at least $\frac{1}{2}(n + 2)^2$.

2. For values of p larger than $\frac{1}{4}(n - 1)(n - 3)$ the preceding method will not apply, as no space curve can have a larger genus, but in this case that of Jonquières ‡ can be readily extended to define the desired curves. Let c_l, c'_l, c''_l be three curves belonging to a net having m base points in common. The curve

$$c_l c''_l \phi_{n-2l} - c'^2_l \psi_{n-2l} = 0$$

will then have these m base points for double points and the constants in ϕ_{n-2l}, ψ_{n-2l} can be so chosen that it will have no others.

* "Sur la classification des courbes gauches algébriques," *Jour. de l'Ecole polytech.*, vol. 52 (1882), pp. 1-200. See page 27.

† *l. c.*, p. 126.

‡ *Mém. de l'Acad. de Paris*, vol. 16 (1858).

The maximum value of m is $l^2 - l + 1$,* and the largest value of l is the largest integer in $\frac{1}{2}n$. When n is even, say $2l$, these curves are already included among those treated before for $r = \frac{1}{2}n - 1$. By taking smaller values for l, curves of all the higher genera can be constructed.

3. When n is odd, say $n = 2l + 1$, the lowest genus obtainable directly is $l^2 - 1$, while the highest one from the space curve is $l^2 - l$. By passing a pencil of c_{l+1} through the l^2 base points of a pencil of c_l, the lower genus can be obtained, and the higher one by projection. To obtain the intermediate cases, pass two c_l through $l - 1$ points on a fixed straight line c_1. Through κ of the $l^2 - l + 1$ remaining points of intersection, the same $l - 1$ points on c_1, and two other points on c_1 pass a pencil of c_{l+2}. The two pencils are now to be made projective in such a way that corresponding curves intersect on c_1. The locus of the point of intersection will be a c_{2l+2}, but c_1 will be a factor. The remaining c_{2l+1} will have $\kappa + l - 1$ double points, and no other singularities. κ can have any value from 0 to $l^2 - l + 1$. This completes the solution of the problem.†

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ON PERIODIC LINEAR SUBSTITUTIONS WHOSE COEFFICIENTS ARE INTEGERS.

BY DR. ARTHUR RANUM.

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1. THE object of this note is to call attention to one or two theorems that follow easily from the results of my paper in this BULLETIN, April, 1907, volume 13, pages 336–345, by taking into account a theorem of Minkowski's given in *Crelle*,

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*C. Kupper: "Ueber das Vorkommen von linearen Schaaren ..." *Sitzungsbericht der Böhm. Gesell.*, Prag, 1892, pp. 264–262. In my article "On curves having a net of minimum adjoint curves," BULLETIN, vol. 14, page 70 (1907) I showed how such a net of curves can be actually constructed by rational operations.

† Since the basis points are not independent, Cayley's theorems regarding the configuration of residual points of intersection do not apply. See Küpper, "Bestimmung der Maximalbasis B für eine irreducible μ-fache Mannigfaltigkeit von Curven $n$ter Ordnung," *Monatshefte für Math. u. Physik*, vol. 6, (1895), pp. 5–11, and my own paper, "On birational transformations of curves of high genus," *Amer. Jour. Math.* vol. 30 (1908), pp. 10–20.