THE FIFTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE fifteenth summer meeting of the Society was convened at the University of Illinois on Thursday and Friday, September 10–11, 1908.   The scientific proceedings were completed in four full sessions, on Thursday and Friday mornings and afternoons.   The social gathering and dinner on Thursday evening at the University Club afforded much pleasure and satisfaction to the members present.

The first session was opened with an address of welcome by Professor Townsend on behalf of the University of Illinois. Owing to the absence of Professor White, President of the Society, all sessions were presided over by Vice-President Professor G. A. Miller, except on two occasions while Professor Miller was presenting his own papers, when Professors Van Vleck and Ziwet respectively were called to the chair.   At the close of the Friday morning session resolutions were adopted expressing the Society's appreciation of the generous hospitality of the University of Illinois and its officers.

The total attendance at the various sessions was over fifty, including the following thirty-eight members of the Society:

Professor L. D. Ames, Professor G. A. Bliss, Dr. R. L. Börger, Professor W. D. Cairns, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. E. L. Dodd, Professor L. W. Dowling, Dr. O. Dunkel, Professor C. Haseman, Professor E. R. Hedrick, Mr. T. H. Hildebrandt, Professor C. N. Haskins, Dr. L. Ingold, Professor O. D. Kellogg, Professor H. W. Kuhn, Dr. E. B. Lytle, Dr. W. R. Longley, Professor G. A. Miller, Professor Max Mason, Dr. C. N. Moore, Professor H. B. Newson, Dr. L. T. Neikirk, Mr. E. W. Ponzer, Professor W. H. Roever, Professor H. L. Rietz, Professor D. A. Rothrock, Professor J. B. Shaw, Professor H. E. Slaught, Mr. L. L. Silverman, Professor E. J. Townsend, Professor J. H. Tanner, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor J. W. Young, Professor A. Ziwet.   Professor M. Abraham, of the University of Göttingen, Germany, was also present.

At the meeting of the Council on Thursday evening, the following persons were elected to membership in the Society : Dr.

G. G. Chambers, University of Pennsylvania; Dr. G. M. Conwell, Yale University; Mr. F. F. Decker, Syracuse University; Mr. W. F. Ewing, California Polytechnic School; Professor A. E. Haynes, University of Minnesota; Mr. P. H. Linehan, College of the City of New York; Mr. H. F. MacNeish, University High School, Chicago; Mr. Lewis Omer, High School, Oak Park, Ill.; Professor J. C. Stone, State Normal College, Ypsilanti, Mich. Six applications for membership in the Society were received. The total membership is now 598.

The following papers were read at the summer meeting:

(1) Professor EDUARD STUDY: " Zur Differentialgeometrie der analytischen Curven."

(2) Professor L. E. DICKSON : " On definite forms in a finite field."

(3) Professor G. A. MILLER : " Answer to a question raised by Cayley as regards a property of abstract groups."

(4) Mr. ARNOLD DRESDEN : " The second derivatives of the extremal integral."

(5) Dr. O. E. GLENN : " On the decomposition of forms into quadratic factors."

(6) Dr. LOUIS INGOLD : " On the Kowalewski integral."

(7) Professor L. D. AMES : " A method for the approximate solution of n equations in n unknowns."

(8) Professor OSKAR BOLZA : " Heinrich Maschke : his life and work."

(9) Professors G. A. BLISS and MAX MASON : " Fields of extremals in space."

(10) Professor J. B. SHAW : " Qualitative algebra."

(11) Professor VIRGIL SNYDER : " Construction of plane curves of given order and genus having distinct double points."

(12) Dr. ARTHUR RANUM : " The divisions of riemannian space into congruent parts."

(13) Dr. ARTHUR RANUM : " Parallelopipeds in lobachevskian geometry."

(14) Dr. F. L. GRIFFIN : " Families of central orbits related to circular trajectories."

(15) Dr. F. R. SHARPE : " The identical relations of the strain and stress components of an elastic solid."

(16) Professor VIRGIL SNYDER : " Surfaces derived from the cubic variety having nine double points in four dimensional space."

(17) Professor L. E. DICKSON : " On Fermat's last theorem."

(18) Professor L. E. DICKSON : " Rational reduction of two quadratic forms " (preliminary communication).

(19) Dr. W. R. LONGLEY : " Note on implicit functions."

(20) Professor E. B. WILSON : " Note on statistical mechanics."

(21) Professor E. J. WILCZYNSKI : " A projective generalization of Meusnier's theorem."

(22) Professor E. R. HEDRICK : " On the convergence of the jacobian."

(23) Professor EDWARD KASNER : " Conformality in connection with functions of two complex variables."

(24) Professor G. A. MILLER : " Groups with regard to modular systems."

(25) Professor J. W. YOUNG : " Two-dimensional chains and the classification of complex collineations in a plane (second paper)."

(26) Mr. J. H. MACLAGAN-WEDDERBURN : " On the direct product in the theory of finite groups."

(27) Professor W. B. FITE : " The class of a group all of whose operations except identity are of order three."

(28) Dr. C. N. MOORE : " The summability of the developments in Bessel functions, with applications."

(29) Professor E. J. TOWNSEND : " Interchange of order of differentiation."

(30) Professor H. B. NEWSON : " On characteristic equations."

(31) Professor L. W. DOWLING : " The arrangement of the real branches of a plane sextic curve."

(32) Professor C. N. HASKINS : " On the second law of the mean."

(33) Dr. L. I. HEWES : " Necessary and sufficient conditions that an ordinary differential equation of the first order and nth degree shall admit a continuous conformal group."

In the absence of the authors, the papers of Professor Study, Professor Dickson, Mr. Dresden, Dr. Glenn, Professor Snyder, Dr. Ranum, Dr. Griffin, Dr. Sharpe, Professor Wilson, Professor Kasner, Mr. Maclagan-Wedderburn, and Professor Fite were read by title.   Professor Bolza's memorial of Professor Maschke was read by Professor Bliss.   The papers of Professor Dickson and Professor Fite were commented upon by Professor

Miller, and the main results of Mr. Dresden's paper were presented by Professor Bliss.

Professor Snyder's first paper appeared in the October BULLETIN. Professor Miller's first paper and Professor Bolza's memorial of Professor Maschke are included in the present number of the BULLETIN. Professor Study's paper will appear in the *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. Defining a definite form to be one which represents only squares in a given field, Professor Dickson determines all m-ary definite forms of degree < 8 in each Galois field of order p^n, $p > 2$. It is shown that every definite quartic form is formally a perfect square. A like result holds for definite sextic forms when $p^n \geqq 13$; but for $p^n \leqq 11$ there occur additional types. The paper has been offered to the *Transactions*. The simplicity of the modular types is in marked contrast to results obtained in the algebraic theory by Hilbert, *Acta Mathematica*, volume 17, page 169.

4. In Mr. Dresden's paper the extremal integral is used for the derivation of necessary conditions for a minimum of the integral

$$\int_{t_1}^{t_2} F(x, y, x', y')dt$$

in the case of one or two variable endpoints and for the case of "discontinuous solutions." The second derivatives of the extremal integral are computed in order to discuss the second variation of the definite integral in these different cases. The results for the cases of one or two variable endpoints are identical with those previously obtained by Bliss. In the problem of the "discontinuous solutions" the conclusion is, in appearance, directly in contradiction with results established by Carathéodory and Bolza. This contradiction is however shown to exist in appearance only, by means of a relation between Weierstrass's E-function and Carathéodory's invariant Ω, a relation which furthermore makes possible a simple proof of a theorem previously proved by the last-named author.

5. The decomposition of ternary and quaternary forms into linear factors has been studied by Junker incidentally in a me-

moir on symmetric functions in the *Mathematische Annalen*, volume 45.   In Dr. Glenn's paper the methods of Junker are extended and supplemented to construct the theory of decomposition of forms into quadratic factors.   The first part of the paper is devoted to the relations which exist among the coefficients of a form in order that it degenerate into such factors. In the second part the original form is assumed to be degenerate, and the component quadratic forms completely determined.   The paper forms a new section of an extended investigation on degenerate curves and surfaces, previously communicated to the Society.

6. G. Kowalewski, in his paper " Ueber den zweiten Mittelwertsatz der Integralrechnung " * has introduced a generalization of the Riemann integral.   The notion involves two functions $u(x)$, $v(x)$ defined in an interval a - b.   This interval is divided into subintervals by the points $x_0 = a$, x_1, x_2, $\cdots$, $x_n = b$. The sum $\sum_1^u (v(x_i) - v(x_{i-1}))u(\xi_i)$ is formed, where ξ_i is any value of x in the ith interval.   If this sum approaches a unique limit when u increases indefinitely in such a way that the greatest subinterval approaches zero, this limit is denoted by $\int_a^b udv$.   This clearly coincides with $\int_a^b uv'dx$ whenever the latter has any meaning.

In the article referred to it is shown that $\int_a^b udv$, in the above sense, exists, 1) if u is continuous and v is of limited variation ; 2) if u is integrable and v is a definite integral function.

The principal results of Mr. Ingold's paper follow :

(1) A necessary and sufficient condition for the existence of the Kowalewski integral is that it be possible to make $\Sigma \omega_i u |\Delta_i v|$ arbitrarily small by making the largest subinterval small enough, $\omega_i u$ being the oscillation of u in the ith interval and $\Delta_i v$ standing for $v(x_i) - v(x_{i-1})$.

(2) If u is integrable, it is sufficient for the existence of $\int_a^b udv$ that v be continuous and $\Delta v/\Delta x$ be bounded.   This last condition on v, however, may be waived in neighborhoods about points of a reducible set.

(3) The elementary formulas for integration hold when the variable is replaced by a continuous function if the integral is then regarded as a Kowalewski integral.

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* *Math. Annalen*, vol. 60, pp. 151–156.

7. Professor Ames gives a method for the approximate solution of n equations in n unknowns under very broad restrictions on the functions involved. The method is a generalization of Newton's method in one unknown and is based on Taylor's theorem. A rough guess is made as to the solution, and successive approximations are obtained after the manner of Goursat's proof for the existence of implicit functions. A geometric interpretation is suggested in the case of two equations. A convergence test analogous to Goursat's is obtained, which can be applied after a sufficient number of approximations have been made, and which yields a limit of error for the root. While the work is analogous to Goursat's, and can be put in much the same form, the problem is essentially different, in that he assumes the existence of one point which satisfies the given equations, and that is what is sought in the present case. It is also found more convenient to handle the equations directly along the lines of Newton's method rather than to make Goursat's somewhat arbitrary transformation. The process frequently converges rapidly even though the first guess is far from a root. If the equations are algebraic the method can be carried out by synthetic division in a form which is a precise generalization of Horner's process, though this is probably not advisable in general.

9. The minimizing space curves for the integral

$$J = \int F(x, y, z, x', y', z')dt$$

form a four-parameter family. Of special interest are the two-parameter families which pass through a given point, or are cut transversely by a given curve or surface. Such families occur when the integral J is to be minimized with respect to curves which join two fixed points, or which have one end point fixed while the other is free to vary on a fixed curve or surface. In the study of the sufficient conditions which insure a minimum for the integral J under these conditions, it is essential to know that such two-parameter families fill out simply a portion of space about the point through which they all pass, or about the curve or surface by which they are cut transversely. The minimizing curves, the extremals, are said to form a field in the region for which this property holds. The field proofs for the analogous cases in the plane have already

been made. (See Bolza, Vorlesungen über Variationsrechnung, page 249 ; or Lectures on the calculus of variations, page 175 ; Bliss, " The construction of a field of extremals about a given point," BULLETIN, volume 13 (1906), page 321. See also Bolza, " Existence proof for a field of extremals tangent to a given curve," *Transactions of the American Mathematical Society*, volume 8 (1907), page 399.) It is shown in the paper of Professors Bliss and Mason that the extremals through a fixed point, or the extremals to which a given curve is transversal, form a field. In these cases the functional determinant of the equations to be solved vanishes at the point, or along the curve in question. By a transformation of the variables the equations are reduced to a set whose functional determinant does not vanish, and the solution is then obtained by the aid of the theory of implicit functions.

10. Professor Shaw's paper deals primarily with the various forms of products in qualitative algebra, by which term is meant general algebra, universal algebra, multiple algebra, and the like. The different kinds of products are classified and examined in turn. Various formulas are deduced for the more important forms.

12. Corresponding to every finite discontinuous group of movements of order m in euclidean or non-euclidean geometry, there is a division of space into m congruent parts. In the case of riemannian space the groups were determined by Goursat and found to be of fifty-one distinct types. In the present paper Dr. Ranum derives the corresponding divisions of space. The boundaries between them are taken to be Clifford surfaces of radius $\frac{1}{4}\pi$. Solids having many of the properties of euclidean parallelopipeds play an important rôle. Study's remarkable correspondence between straight lines of riemannian space and pairs of points lying on a pair of spheres in euclidean space is utilized to great advantage.

13. In his second paper Dr. Ranum gives a classification of most important solids in lobachevskian space whose properties are analogous to those of parallelopipeds in euclidean space and which degenerate into parallelcpipeds when the space becomes euclidean.

14. Dr. Griffin considers the circle as a central orbit, its center not coinciding with the center of force, and shows that

all families of such orbits are characterized by two invariant properties. He then discusses the question of the most general laws of central force admitting families of trajectories possessing one or the other of these properties; and, in particular, points out for the more usual laws the families in question.

15. The 6 components of strain of an elastic solid are linear functions of the derivatives of the 3 displacements. Six identical relations between the components were first given by St. Venant. Dr. Sharpe shows that there are 15 such relations, of which only 3 are independent. The 6 components of stress are linear functions of the 6 components of strain and are therefore subject to 15 similar relations. In the case of equilibrium under surface tractions alone there are 3 additional equations of equilibrium. Six identical relations between the stress components were first given by Beltrami. It is shown in this paper that the number of relations between any of the 6 components is equal to their number and that all but one of these are independent.

16. In the *Turin Memoirs* for 1888 Professor Segre studies a large number of forms of cubic varieties in four dimensional spaces. He also discusses the apparent contour in ordinary space, and various congruences having the contour for complete focal surface. The treatment is entirely synthetic, and no case is treated in detail. Professor Snyder considers the case of the cubic variety having nine double points from the analytic standpoint, and derives two new forms of (3, 3) congruences. The first has a focal surface with two plane cubic cuspidal curves and three triple planes, each containing six double points. The other case presents a family of congruences all contained in a fixed linear complex. The focal surface of order and class six is complete focal surface for six such congruences. It has a space sextic for cuspidal curve, nine double points, and nine double planes. Through each point pass five planes and in each plane lie five points.

17. Fermat's equation $x^n + y^n + z^n = 0$ is proved to be impossible in integers prime to n for every odd prime $n < 7000$. The method is based upon the corresponding congruence modulo p, where p is a prime of the form $mn + 1$. For each $m < 74$ and for $m = 76$ and 128, all the primes p are determined for which the congruence has solutions. The results are

given in two papers now appearing in the *Messenger of Mathematics*. In a subsequent paper, offered to *Crelle's Journal*, Professor Dickson establishes by means of the cyclotomic theory the theorem that, when m exceeds a certain cubic function of n, the congruence has a set of solutions prime to p. This theorem is in contrast to the known result, admitting of immediate proof, that, for a fixed value of m prime to 3, the congruence has no solutions except when n has one of a finite set of values.

19. In the case of one equation $f(x, t) = 0$ (where f is expansible in positive integral powers of x and t and $f(0, 0) = 0$), defining a function x of one independent variable t, the condition that the jacobian shall be different from zero (for $x = t = 0$) is a sufficient condition that the function x shall be defined uniquely as a series in positive powers of t vanishing with t. If $f(x, t)$ does not contain t as a factor, the condition is also necessary for a unique solution.

In the case of two equations $f(x, y, t) = 0$, $g(x, y, t) = 0$ (where f and g are expansible in positive integral powers of x, y, t and $f(0, 0, 0) = 0$, $g(0, 0, 0) = 0$), defining x and y in terms of t, the condition that the jacobian shall be different from zero (for $x = y = t = 0$) is a sufficient condition that x and y shall be defined uniquely as series in positive powers of t vanishing with t. Supposing that neither $f(x, y, t)$ nor $g(x, y, t)$ contains t as a factor, the condition is however not necessary for a unique solution. It is shown in Dr. Longley's note that the condition that the jacobian shall be different from zero is a special case of more general conditions upon f and g in order that x and y shall be defined uniquely as series in positive powers of t vanishing with t. For certain special cases in which the jacobian vanishes, the general conditions furnish practical criteria which are analogous to the ordinary jacobian condition. The general method can be extended to any number of equations.

20. Professor Wilson discusses the analogy between hydromechanics and statistical mechanics. In case the number of degrees of freedom of the dynamical system which is to be treated statistically is $n = 1$, tolerably satisfactory results are obtainable; when $n > 1$ the equations of the fictitious fluid which represents the statistical problem must in general be more complicated than the hydromechanic type. A discussion of the physical dimensions of the quantities arising in connec-

tion with the analogy discourages the attempt to find a medium which shall satisfactorily represent the problem.

21. Meusnier's theorem may be stated as follow : Consider those sections of a surface made by planes which pass through a fixed point and a fixed tangent of the surface. The circles which osculate these plane sections at the fixed point common to all of them generate a sphere.

In Professsor Wilczynski's generalization osculating conics, *i. e.*, conics having fourth order contact with the plane sections, take the place of the osculating circles. Their locus is a quadric surface.

The detailed discussion of special and exceptional cases gives rise to some interesting results. The author also considers the surface which is the locus of the osculating conics of those plane sections which contain the directrix of the second kind of the surface point considered. If we think of this directrix as a projective substitute for the normal, this consideration may be characterized as a projective generalization of Euler's theorem.

22. Criteria for differentiation of a series of functions of a real variable term by term have been a subject of serious study and are now well known in several forms of varying sharpness. In general they require that the series which results upon attempted differentiation should converge uniformly, with a further condition upon the original series which essentially requires its uniform convergence, though this is redundant with the previous requirement and may be partially suppressed in the statement of the theorem.

Similar criteria for the term by term differentiability of series of functions of a complex variable are usually stated, but the conditions are apparently smaller on account of the usual assumption that the function is analytic.

Professor Hedrick's paper considers in a similar manner the convergence of the jacobian of a set of transformations which approach a limiting transformation, or, what is the same thing, of a set of unrestricted functions of a complex variable which approach a limiting function. It is shown that the jacobian of the limiting transformation is the limit of the sequence of jacobians, if the series which arise from partial differentiation of the functions defining the sequence of transformations each converge uniformly. Moreover, it is shown that if the transformations

converge uniformly, and if the jacobians converge uniformly, the jacobian of the limiting transformation is the limit approached by the jacobians.

These statements appear in more elegant form with the notation of the theory of functions of a complex variable, in its nonspecialized form. Theorems analogous to each of the preceding are stated for an uncountable set of functions which cluster about a given function.

23. In the theory of functions of two complex variables $z = x + iy$, $w = x' + iy'$, the transformations of importance are of the form $Z = \phi(z, w)$, $W = \psi(z, w)$, where ϕ and ψ are general analytic functions. In the four-dimensional space with cartesian coordinates x, y, x', y', these transformations are not conformal, and Poincaré in his recent paper in the *Palermo Rendiconti* therefore employs (provisionally) the term regular. Professor Kasner obtains several geometric characterizations of the regular transformations. Two linear systems of planes are converted into themselves and only the angles situated in these planes are left invariant. The family of surfaces affected conformally is easily derived. The simplest characteristic invariant (pseudo-angle) is connected with the intersection of a line with a three-dimensional variety. In conclusion certain finite subgroups are considered : the 9-parameter conformal group, the linear, and the linear fractional groups.

24. Professor Miller's second paper is composed of two distinct parts. The first is devoted to a list of all the possible modular systems of the form [mod p, $\phi(x)$] by means of which the abstract abelian groups whose orders do not exceed 12 may be represented as congruence groups. This part may be regarded as complementing, along a certain line, the articles of Kempe and Cayley in which all the possible abstract groups whose orders do not exceed 12 are considered at some length. The second part is devoted to a few general theorems relating to the representation of abstract groups in regard to Kronecker's modular systems. Among the theorems proved in this part are the following: It is not possible to represent every abstract abelian group as a congruence group in regard to some modular system. The lowest group which does not have the property that it may be represented by a complete set of prime residues with respect to a modular system is of order 5. If $\phi(x)$ is the

product of k distinct irreducible functions $(\phi_1, \phi_2, \cdots, \phi_k)$, then the residues which are prime mod p, $\phi(x)$ constitute the direct product of the groups with regard to the separate modular systems p, ϕ_1; p, ϕ_2; $\cdots$; p, ϕ_k. The number of the invariants equal to p^a in the group formed by the prime residues mod p, x^n is $\epsilon'[n/(p^{a-1})] - 2\epsilon'[n/(p^a)] + \epsilon'[n/(p^{a+1})]$, where $\epsilon'[x]$ is the largest integer which is less than x. The last theorem is regarded as especially interesting in view of the fact that it exhibits the marked difference between the properties of these groups and those formed by the numbers which are prime to p^m, when they are combined by multiplication and the products are reduced mod p^m.

25. A complex projective space S_n of n (complex) dimensions is defined analytically as the totality of points $(x_0, x_1, \cdots, x_n)$, where the homogeneous coordinates x_i are any complex numbers not all zero. A projective transformation in this space is represented by a linear homogeneous transformation on the x_i with complex coefficients. The totality of points whose coordinates are real form a subspace R_n of n (real) dimensions in S_n. Every subspace of S_n which is obtained by subjecting R_n to a collineation in S_n is called an n-dimensional chain in S_n, or more briefly an n-chain.

In a paper presented to the Society at its last April meeting in New York (abstract in BULLETIN, volume 14 (June, 1908), page 411), Professor Young developed certain properties of the two-chains of a complex plane and applied the results to the classification of the collineations in a plane " with respect to reality." The present paper continues the investigation there begun by effecting a classification "with respect to reality " of the continuous groups of collineations in a plane. The object is to enumerate all types of such groups which leave a two-chain invariant ; every such group can evidently be transformed into one in which all the coefficients of every collineation are real. Two groups leaving the same two-chain invariant are regarded as equivalent, if and only if they can be transformed into each other by a collineation leaving the same two-chain invariant. The treatment of the problem is very largely synthetic and on the basis of the results of the first paper and with the help of the known lists of the groups of collineations in a plane (cf. Newson, *American Journal*, volume 24, page 169, and Meyer, Chicago Congress Papers, page 188) offers little

difficulty. The author finds seven types of groups in addition to those obviously obtainable from Newson's list by restricting the invariant elements to be real, or from Meyer's list by restricting the coefficients to real values. That the enumeration is complete rests on Lie's theorem that every continuous group with real coefficients gives rise to a complex group of the same number of parameters, if the parameters are allowed to assumed complex values.

26. Mr. Wedderburn shows that, if a group be expressed in two distinct ways as a direct product of prime factors (a group being prime if its only factors are itself and the identity), the factors in one series are simply isomorphic with the factors, taken in a suitable order, in the other series. Further properties of such series are established. The paper will be offered to the *Transactions*.

27. The main result in Professor Fite's paper is that a group all of whose operations, except identity, are of order three is abelian, metabelian, or of class three. It is also shown that such groups of class three exist.

28. In this paper Dr. Moore discusses the summability of the development of an arbitrary function in terms of Bessel functions. Sufficient conditions upon the arbitrary function are obtained that the development should be summable * to the value of the function at every point of the interval $0 < x < 1$ at which the function is continuous and should be uniformly summable throughout any sub-interval of an interval in which the function is everywhere continuous. The convergence factors which occur in connection with these developments in the investigation of problems in the flow of heat are then discussed and the theorems established are used in showing that the formal results obtained by the methods of Fourier really furnish the desired solution of the problems.

29. Professor Townsend called attention to the fact that the conditions usually given for the interchange of the order of differentiation involve the continuity with respect to the two variables taken together of the partial derivatives concerned. He then showed that by a slight modification of the proof given

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* The definition of a summable series here adopted is the same as that used by the writer in a previous paper (Cf. *Trans. Amer. Math. Soc.*, vol. 8 (1907), p. 299).

by Schwarz* it can be shown that whenever $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x \partial y$ exist and are continuous with respect to each variable separately, the partial derivative $\partial^2 f/\partial y \partial x$ exists also and is equal to $\partial^2 f/\partial x \partial y$.

30. In abstract Professor Newson's paper is as follows : Let T be a linear transformation in n variables, thus :

$$\rho x_i' = a_{1i}x_1 + \cdots + a_{ni}x_n \quad (i = 1, 2, \cdots, n).$$

Let the determinant of T be such that T represents a collineation of the most general type in S_{n-1}, a space of $n-1$ dimensions. It leaves invariant n linearly independent points A_i, whose coordinates may be represented by A_{1i}, A_{2i}, $\cdots$, A_{ni} $(i = 1, 2, \cdots, n)$. Let λ_i $(i = 1, 2, \cdots, n)$ be n independent constants such that λ_i/λ_1 $(i = 2, \cdots, n)$ are the cross ratios of the $n-1$ one-dimensional projective transformations along the invariant lines $A_1 A_i$ $(i = 2, \cdots, n)$. The linear transformation T may now be written in the explicit normal form, as follows :

$$T : \qquad \rho x_i' = \begin{vmatrix} x_1 & \cdots & x_n & 0 \\ A_{11} & \cdots & A_{n1} & \lambda_1 A_{i1} \\ \vdots & & \vdots & \vdots \\ A_{1n} & \cdots & A_{nn} & \lambda_n A_{in} \end{vmatrix}$$

The determinant of the normal form of T is $D = \lambda_1 \cdots \lambda_n \Delta^n$, where $\Delta = |A_{ij}|$. The characteristic equation of the normal form of T breaks up into linear factors, thus :

$$\prod_1^n [\rho + (-1)^n \lambda_i \Delta] = 0.$$

If ρ/Δ be replaced by ρ', the roots of the characteristic equation in ρ' are $(-1)^{n+1} \lambda_i$ $(i = 1, \cdots, n)$.

31. Professor Dowling's paper is in abstract as follows : The equation of any proper sextic having nine double points may be reduced to the form $\lambda S + U^2 = 0$, where $S = 0$ is one such sextic and $U = 0$ is a cubic passing through the nine double points. Elliptic sextics having nine crunodes are of three forms : a single even branch crossing itself nine times, two odd branches, or two even branches.

Choosing λ in such a way that the sextic $\lambda S + U^2 = 0$ pos-

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* Abhandlungen, vol. II, p. 275.

sesses the maximum number of connected loops, i. e., loops within which $\lambda S + U^2$ has the same sign (this maximum number is nine), the sextic $\epsilon(\lambda S + U^2) = V^2$, for proper values of ϵ and when U crosses the nine loops of $\lambda S + U^2 = 0$, consists, in general, of nine ovals within the loops of $\lambda S + U^2 = 0$. But when U is unipartite and V is bipartite the sextic $\epsilon(\lambda S + U^2) = V^2$ has in addition two ovals, one within and one without the oval of V.

This method of building up the general sextic from a crunodal elliptic sextic prevents the escape of ovals from the loops of the elliptic sextic and at the same time insures the maximum number of branches which a sextic may have. It moreever shows the necessity for the nested branches of the sextic.

32. Professor Haskins's note points out that the methods used in the proof of the second law of the mean by application of the fundamental theorem of Fourier's constants * apply to the intervals of Lebesgue as well as to those of Riemann.

33. If we write an ordinary differential equation of the first order and nth degree in the minimum coordinates $u = x + iy$, $v = x - iy$, it has the form

$$\pi^n + a_1\pi^{n-1} + a_2\pi^{n-2} + \cdots + a_i\pi^{n-i} + \cdots + a_{n-1}\pi + a_n = 0$$

(1)
$$(\pi = dv/du).$$

A necessary and sufficient condition that this differential equation admit the conformal infinitesimal transformation

$$U'f: \qquad \phi(u)\frac{\partial f}{\partial u} + \psi(u)\frac{\partial f}{\partial v} + \pi\left[\psi'(v) - \phi'(u)\right]\frac{\partial f}{\partial \pi}$$

is found from the invariants. There are $n - 1$ invariants involving the coefficients a and these lead to $n - 1$ involving the first partial derivatives of a. For the above property it is then necessary and sufficient that these $2n - 2$ invariants be functions of the same harmonic function.

The theory is applied to the equation of the second degree $\pi^2 + a_1\pi + a_2 = 0$ and to a concrete example of this type.

<div align="right">H. E. SLAUGHT,
Acting Secretary.</div>

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* Cf. Haskins, *Annals of Math.*, ser. 2, vol. 9 (1908), p. 173.