GOURSAT'S COURS D'ANALYSE.

Cours d'Analyse mathématique. By ÉDOUARD GOURSAT, Professeur à la Faculté des Sciences de Paris. Vol. II, *Theory of Analytic Functions; Differential Equations, Total and Partial; Elements of the Calculus of Variations.* Paris, Gauthier-Villars, 1905. 8vo, pp. vi + 640.

THE second and last volume of Professor Goursat's treatise on the elements of analysis is devoted to the theory of functions of one and of several complex variables, pages 1–305, and to differential equations, total and partial, pages 306–589, concluding with a chapter of forty-two pages on the calculus of variations. Like the first volume, the book is characterized by the vital relation of the choice of material and the mode of treatment to the analysis of the present time. It is the product of a man whose contributions to analysis are of value and who wishes to give to his students the tools he has found to be useful. The procedure that is simplest in practice is given the most prominent place in the presentation, and the main theorems of the subject treated are set in a strong light. On the other hand, a wide range of topics is treated, an outline of a whole theory being sometimes given well on in a chapter as an application of the foregoing principles. When this is done, the author shows good judgment in what he expects of his readers. It is reasonable to leave to the student at this stage the details of proofs which follow general well-defined lines, provided that the framework is so constructed that he can see what is required. The tact which the author displays in this regard makes the book a serviceable one for advanced students to read by themselves and to report on to their university teacher.

The volume begins with an introductory chapter of seventy-four pages, Chapter XIII, on analytic functions of a complex variable, in which both single and multiple-valued functions are treated, the elementary functions being defined in a natural manner for complex values of the argument. The evaluation of integrals of the type

$$\int R(x)dx, \quad \int R(\sin x, \cos x)dx,$$

where R is a rational function, is taken up, and the subjects of infinite series and products, conformal mapping both of plane and of curved surfaces, with applications to Mercator's chart and stereographic projection and to the transformation of an anchor ring on a rectangle, and families of isothermals, are set forth. In the twenty-five examples that conclude the chapter a number of properties of linear transformations of a complex variable are brought out.

Chapter XIV is devoted to an exposition of Cauchy's theory of functions, with applications to singular points and to Weierstrass's theorems about series of analytic functions. Lagrange's and Burmann's series are treated, the hyperelliptic integrals are studied with reference to questions of periodicity, and a proof that the ratio of the moduli of periodicity of an elliptic integral of the first kind cannot be a real quantity is given, by the aid of Weierstrass's law of the mean for complex integrals, without breaking the integral up into its real and its pure imaginary part. Applications to spherical harmonics appear repeatedly.

Chapter XV, on single-valued functions, contains the theorems of Weierstrass and Mittag-Leffler. The development of $\cot x$ is obtained by the method of residues, the contour of integration being a rectangle. Fifty-eight pages are then devoted to the theory of doubly periodic functions and their inverse, with applications to algebraic curves of deficiency unity.

Chapter XVI, on analytic continuation, and Chapter XVII, on analytic functions of several variables, conclude the part of the volume devoted to the theory of functions. The definition of an analytic function of two independent variables, given on page 264, is as follows: Let A and A' be two regions of the z-plane and the z'-plane respectively and let $f(z, z')$ be uniquely defined for each pair of values z, z' belonging to these regions. If $f(z, z')$ is continuous and each partial derivative $\partial f/\partial z$, $\partial f/\partial z'$, exists at each such point (z, z') — *i. e.*, if each of the ratios

$$\frac{f(z+h, z') - f(z, z')}{h}, \qquad \frac{f(z, z'+k) - f(z, z')}{k}$$

remains finite and approaches a limit as h and k approach 0, — then $f(z, z')$ is said to be an analytic function of the two independent variables z and z'. Since the publication of the book the investigations of Hartogs in the 62d volume of the *Mathematische Annalen* have shown that the condition of continuity is superfluous, being a consequence of the other conditions. Abel's theorem in the theory of the abelian integrals and Darboux's extension of Lagrange's series find a place in the latter chapter.

Passing now to the subject of differential equations, we find in Chapter XVIII an account of the elementary methods for solving total differential equations, or of reducing such an equation in certain cases to equations of lower order. Some properties of Riccati's equation are discussed in § 369, and in § 377 Euler's equation for the elliptic integrals is treated by means of Abel's theorem.

Chapter XIX, on the existence theorems, is exceedingly well done. First comes the proof of the existence of a solution of a system of ordinary differential equations of the first order and of a similar system of partial differential equations, the method being that of power series and *fonctions majorantes*. The proof in the latter case is especially simple and well arranged. Thus in three paragraphs, §§ 383, 384, and 386, all about existence theorems is contained which the reader needs for the cases of chief interest in what follows. But Cauchy's proof by means of a broken line, and the method of successive approximations are given in full. In § 387 the development of the fundamental properties of the solution of a system of ordinary differential equations begins, — properties which in the older treatments like those of Boole and Forsyth (1885) were assumed without proof and which obviously do not hold without restriction, even in simple cases. For example, according to those views, every differential equation of the first order,

$$\frac{dy}{dx}=f(x, y),$$

admits a solution containing an arbitrary constant,

$$y = \phi(x, C),$$

and the particular solution which assumes the value β when $x = \alpha$ can be obtained by solving the equation

$$\beta = \phi(\alpha, C)$$

for C. Furthermore, it was implied or explicitly stated in those earlier days that any function

$$y = \phi(x, C)$$

which is such that the elimination of C between the two equations

$$y = \phi(x, C), \qquad \frac{dy}{dx} = \frac{\partial \phi(x, C)}{\partial x}$$

leads to the given differential equation is *the* solution, *i. e.*, a solution with the above properties. Now, entirely aside from any questions of singular solutions or peculiar functions, this latter theorem is not true. Consider, for example, the differential equation

$$\frac{dy}{dx} = y.$$

A solution of this equation of the above kind is the following:

$$y = \phi(x, C) = e^{x+C}.$$

Moreover, the equation has a solution which vanishes when $x = \alpha$, namely,

$$y = 0.$$

But the constant C cannot be so determined as to yield this solution.

In the early days of modern analysis crinkly curves and uncanny sets of points were most in evidence. But the real contribution of Cauchy and Weierstrass was to furnish the means of obtaining theorems at the threshold of a great subject like differential equations where no theorems of general validity had hitherto existed and where, therefore, no theorem hitherto known could be applied with security in any special case. The French early perceived this phase of progress in analysis, and the book before us is written in this spirit.

I know of no subject, unless it be the study of polynomials and elimination in algebra, that is better suited to give to the student who has just covered the elements of the theory of functions of one or more complex variables, power to apply the theory than a careful working out of the elements of the theory of differential equations, such as is outlined in the chapters before us. I use the word *outlined* advisedly. For while Professor Goursat sets forth in bold relief the main theorems of the theory, he does not in all cases develop the details to the point that the beginner must reach. This is a strong feature of the book for purposes of instruction. For example, the theorems about implicit functions have been carefully developed in their proper place, and they are frequently referred to in these chapters. But it happens time and again that the ultimate formulation is left to the student. And it is for this reason that I included those parts of algebra that deal with linear dependence and elimination, for it is here, where the attention is not distracted by difficulties in the theory of functions, that the student best learns what the problems of elimination really are.*

But I should give a wrong impression of Professor Goursat's work if I should fail to emphasize the wealth of illustration and the skill with which the leading ideas of a whole theory are at times brought out. Take, for example, this same chapter with the arid title: "Existence theorems." Here we find in three paragraphs at the end, entitled respectively: "One-parameter groups," "Applications to differential equations," and "Infinitesimal transformations," the greater part of the fundamental theorems and points of view of the first two thirds of Lie-Scheffer's book on Differentialgleichungen mit bekannten Infinitesimaltransformationen. Moreover, the theorems are here proven or so far formulated that the student can touch bottom. At least the latter statement becomes true if a slight alteration is made, being in substance what is done at the beginning of Lie-Engel, Transformationsgruppen; for thereby some of the difficulties that present themselves at the outset in connection with the conception of a group are satisfactorily met. The suggestion is this, that we restrict ourselves to the neighborhood of one of the points that is to be transformed and to that of the parametric values corresponding to the identical transformation. Thus for the one-parameter group in the plane we should require that the two functions

$$x' = f(x, y, a) \qquad y' = \phi(x, y, a)$$

satisfy the following conditions:

a) They shall both be analytic at a point (x_0, y_0, a_0);

b) If

$$x' = f(x, y, a), \qquad y' = \phi(x, y, a),$$

and

$$x'' = f(x', y', b), \qquad y'' = \phi(x', y', b),$$

then

$$x'' = f(x, y, c), \qquad y'' = \phi(x, y, c),$$

where (x, y) on the one hand, and a and b on the other, are chosen arbitrarily in the neighborhoods of the point (x_0, y_0) and the point a_0, and where, furthermore,

* On this subject see, in particular, Bôcher, Introduction to Higher Algebra, Chaps. III, IV.

$$c = \psi(a, b)$$

is a function analytic in the point $a = a_0$, $b = b_0$;

c) $f(x, y, a_0) = x, \qquad \phi(x, y, a_0) = y,$

where (x, y) is an arbitrary point of the neighborhood of (x_0, y_0);

d) At least one of the derivatives $\partial f/\partial a$, $\partial\phi/\partial a$ shall not vanish at the point (x_0, y_0, a_0).

From b) and d) it follows that

$$\psi(a, a_0) = a, \qquad \psi(a_0, b) = b,$$

as also, in particular, that $\psi(a_0, a_0) = a_0$. And from c) we infer that the jacobian of the transformation,

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix},$$

does not vanish at the point (x_0, y_0, a_0), its value there being unity.

With these restrictions, all the theorems of the text are readily proven *im Kleinen* for the neighborhood of the point (x_0, y_0, a_0). By means of analytic continuation the theorems can then be extended, and thus the difficulties that present themselves at the beginning are reduced to a minimum.*

Chapter XX is devoted to linear differential equations, the equations of classical type being integrated explicitly and the developments of Fuchs in the neighborhood of a regular singular point being deduced. In Chapter XXI non-linear differential equations are studied. This chapter contains a good treatment of singular solutions.

But the author goes too far at times in omitting to state the restrictions which it is intended to impose. For example, consider § 427, which deals with the author's special investigations

*The procedure in the first chapter of Lie-Engel, Transformationsgruppen, is similar, except that there sufficient conditions to insure the existence of the identical transformation and of inverse transformations are not introduced at the outset, this fact being explicitly stated. On the other hand, Lie-Engel are especially careful to call attention to the necessity of employing a subneighborhood, $((x))$, $((a))$ in their notation. The omission of this point by Professor Goursat does not contribute to accuracy or clearness.

on congruences of curves. In order to avoid unnecessary generalities he very properly restricts himself to the case of two polynomials

$$F(x, y, z, a, b) = 0, \qquad \Phi(x, y, z, a, b) = 0.$$

But his arguments do not hold for polynomials, as is seen from the example

$$F(x, y, z, a, b) = (y - a)^2 - bx^2,$$
$$\Phi(x, y, z, a, b) = z - b,$$

both of these polynomials being, moreover, irreducible in the five arguments. It is not obvious what restrictions he means to observe. The above polynomials correspond to the system of differential equations

$$\frac{dy}{dx} = \sqrt{z}, \qquad \frac{dz}{dx} = 0.$$

Chapter XXII contains in sixty-three pages an admirable introduction to the elementary theory of partial differential equations. It is true that the proofs are not always carried down to such a point that the beginner will see how to make contact with his theorems about implicit functions ; but he must be able to make such formulations for himself if he expects to do anything with modern analysis, and here is a good chance for him to learn his trade by working through in detail what is well defined in the text. Differential equations and simultaneous systems of the first order in three variables are studied and the theory is illustrated geometrically. Cauchy's problem and the method of characteristics are discussed, and the chapter closes with a paragraph on the Monge-Ampère equations of the second order.

The last chapter of the book, on the calculus of variations, gives an account of those problems and methods, with the development of which the school of Weierstrass has occupied itself, strong and weak variations, sufficient conditions, the *E*-function, fields of extremals, and Hilbert's invariant integral being the catch-words that suggest the developments.

In these two volumes Professor Goursat has given us a standard work on analysis, one that is modern in all respects, and which will undoubtedly be fully appreciated by students of mathematics and physics.

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