many valuable earlier publications, the author of the present work has won the confidence of the mathematical public, and the usefulness of the present volume will doubtless increase this confidence. It is hoped that it may also inspire others to render such scholarly services even if their accomplishment demands a vast amount of time, care, and patience. The more our science grows the more need there will be for such works, and conversely such works contribute materially towards stimulating real growth.

G. A. MILLER.

Ueber das Wesen der Mathematik. Rede gehalten am 11. März 1908 in der öffentlichen Sitzung der K. Bayerischen Akademie der Wissenschaften von DR. A. VOSS, O. Professor der Mathematik in München. Erweitert und mit Anmerkungen versehen. Leipzig und Berlin, B. G. Teubner, 1908. 98 pp.

ADDRESSING a mixed assembly of scientists, Voss endeavors to answer the questions: What is the nature (Wesen) of mathematics? How does it happen that mathematics is the only science which presents truth in apodictic form? What has the past century accomplished toward the elucidation of the inner structure of mathematics?

Though not answering these difficult questions with the precision some may desire, the address is extremely able and instructive. The annotations occupy about half the space of the book and contain numerous valuable references to the literature of the subject. The author passes in rapid historical review the fundamental concepts of variables, functional dependence, and limits. He points out the triumphs of mathematics during the eighteenth century, a period which culminated with Laplace. This great savant made the well-known utterance, now admitted to be a gross exaggeration: An intelligence to whom at a given moment were presented the conditions of the entire material world would be able by mathematical analysis to survey the entire past and future of the world. In considering the various attempts to define mathematics, Voss passes from the antiquated "science of quantity" definition to the more recent ones based on the consideration of the logical steps involved, namely the definitions of B. Peirce, E. Papperitz, G. Itelson, B. Kempe, M. Bôcher, H. B. Russell, and L. Couturat. Geometry and mechanics belong to applied mathematics. Pure mathematics has come to be considered the science of number (die Wissenschaft von den Zahlen). There follows a sketch of the historical development of the number concept. It is pointed out that the axiom of Eudoxus and Archimedes does not hold for complex numbers. If $a + bi \gtrless c + di$, when $a \gtrless c$, or when $a = c$ and $b \gtrless d$, then not only is b^2i less than $c^2 + di$, but nb^2i is less also, no matter how large a positive integer n may be. Nor does the axiom hold for types of real numbers introduced by Hilbert in his Grundlagen. Voss discusses the nineteenth century researches on the concept of a function of a complex variable, on differential equations, on Cauchy's and Riemann's definitions of an integral, the recent theory of manifoldness and the paradoxes which still encumber it. Toward the close he touches upon geometry, makes war upon Wundt for his critique of non-euclidean geometry, and speaks of the work of Peano and Hilbert on the foundations of geometry. Lastly he raises the question, is mathematics after all not a huge tautology? If everything rests upon number and its assumed laws of combination, then everything is included in the logical unfolding of the postulates, and a powerful intellect should be able to view all consequences at a glance. Poincaré answers the question in the negative, for mathematical reasoning involves a creative power, displayed in the act of the so-called mathematical induction from n to $n + 1$. Fr. Meyer answers that, from even a small number of axioms, one may draw an unlimited number of deductions and the whole science reduces itself to a gigantic game of chess. Voss himself likewise answers the question in the negative, but considers both explanations inadequate in the light of history. In the ability of the human mind to make new experiences, to draw from them general ideas and subject these to mathematical analysis, to numerical treatment, we must seek the steady growth of the science. Thus, the idea of continuity gave rise to irrational number, the idea of motion to the concept of a limit.

We have noticed only a few historical errors. Logarithms were invented in 1614, not 1620 (page 11); the Diophantine symbol of subtraction is given incorrectly (page 11); the principle of local value in writing numbers is ascribed to the Hindus (page 10), but was known earlier by the Babylonians, at least in part. In referring to Babylonian mathematics, Voss fails to mention the important researches of Hilprecht, of the University of Pennsylvania.

On the authority of Moritz Cantor,* Voss ascribes the introduction of the inference by mathematical induction to Pascal (before 1654) and its independent re-introduction to Jacob Bernoulli (1680). This important process is called by Max Simon † the " Bernoullian " or the " Kästnerian " inference. Charles S. Peirce ascribes it to Fermat and calls it the " Fermatian " inference.‡

In view of these conflicting historical statements, it is worth while to inquire more minutely into the origin of the process of mathematical induction. The claim for Kästner hardly needs consideration ; he and other writers used it in the second half of the eighteenth century. Nor can priority be established for Jacob Bernoulli. The claim for Pascal rests upon passages in his Traité du triangle arithmétique, particularly the one relating to a certain proportion : § " Quoique cette proposition ait une infinité de cas, j'en donnerai une démonstration bien courte, en supposant deux lemmes." The first lemma says that the proportion is seen to be true for numbers on the second diagonal (" base ") ; the second lemma asserts, " que si cette proportion se trouve dans une base quelconque, elle se trouvera nécessairement dans la base suivante. D'où il se voit qu'elle est nécessairement dans toutes les bases : car elle est dans la seconde base par le premier lemme ; donc par le second elle est dans la troisième base, donc dans la quatrième, et à l'infini." This process is precisely what is now designated by mathematical induction.

Charles S. Peirce does not cite, in substantiation of his claim, any particular passage from the writings of Fermat, nor does he make any specific reference to them. Peirce describes mathematical induction in the manner current in modern texts and ascribes it to Fermat. We have not been able to find in Fermat's works the process of mathematical induction in its purity. In the famous Fermatian document, discovered among the papers of Huygens in 1879, and entitled " Relation des découvertes en la science des nombres," Fermat describes a method which he calls " la descente infinie ou indéfinie." He says that it was particularly applicable in proving the impossibility of certain rela-

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* Vorlesungen über Geschichte der Mathematik, vol. 2, 2te Aufl., Leipzig, 1900, p. 749.

† Methodik d. elem. Arithmetik, Leipzig und Berlin, 1906, p. 13.

‡ Century Dictionary, Art. " Induction ;" *The Monist*, vol. 2 (1892), pp. 539, 545.

§ Oeuvres complètes de Blaise Pascal, vol. 3, Paris, 1866, p. 248 (*conséquence* XII).

tions, but that he finally succeeded in using the method also in proving affirmative statements. "Si un nombre premier pris à discretion qui surpasse de l'unité un multiple de 4 n'est point composé de deux quarrez, il y aura un nombre premier de mesme nature moindre que le donné; et ensuite un troisième encore moindre, etc., en descendant a l'infini jusques à ce que vous arriviez au nombre 5, qui est le moindre de tous ceux de cette nature, lequel il s'en suivroit n' estre pas composé de deux quarrez, ce qu'il est pourtant d'ou on doit inferer par la deduction à l'impossible que tous ceux de cette nature sont par conséquent composez de 2 quarrez." This famous method, known to Fermat probably as early as 1636 or 1637, contains a recurrent mode of inference, embodying in one step an indefinitely large number of syllogisms. But it is not mathematical induction in its purity; it is such a process applied to a descending order of progression, and appearing as the superstructure to a reductio ad absurdum argument. Moreover, Fermat does not descend in regular progression from n to $n-1$, $n-2$, etc., but leaps irregularly over usually several integers from n to $n-n_1$, $n-n_1-n_2$, etc. Neither Moritz Cantor * nor H. G. Zeuthen † calls this mathematical induction or "vollständige Induktion."

Even if this term be used in a sense broad enough to include the mode of inference described by Fermat, should it be called the "Fermatian inference"? We think not. In the first place his Relation did not become general property of mathematicians until 1879. Fermat's leaning was toward keeping his methods secret. He influenced mathematicians in the theory of numbers by the *results* he reached, rather than by the methods he made known. In the second place, the process in question was used 300 years earlier by Campanus in his edition of Euclid (1260). Campanus proves the irrationality of the golden section. His argument is reproduced by Genocchi and Cantor in algebraic notation thus: ‡ If $(x_1+x_2):x_1=x_1:x_2$, then also $x_1:x_2=x_2:(x_1-x_2)$ or, putting $x_1-x_2=x_3$, where $x_1>x_2$, we have $(x_2+x_3):x_2=x_2:x_3$, where $x_2>x_3$. In the same way we get $(x_3+x_4):x_3=x_3:x_4$, where $x_4=x_2-x_3<x_3$, and so on indefinitely. Since there is only a finite number of positive integers less than x_1+x_2, the first proportion is incorrect; x_1 and x_2 can not be integers; q. e. d.

* Vorlesungen über Geschichte der Mathematik, vol. 2, 2te Aufl., Leipzig, 1900, pp. 778–780.

† Geschichte der Mathematik im XVI. und XVII. Jahrhundert, deutsche Ausg. v. Raphael Meyer, Leipzig, 1903, pp. 157–162.

‡ Cantor's Vorlesungen, vol. 2, 2. Aufl. 1900, p. 105.

The theory of numbers received much attention from the Hindus and Greeks, and it would be surprising if the recurrent mode of inference were not found among them. It does not appear, however, strictly in the form of the modern mathematical induction and free from entanglement with other processes. The "cyclic method" of Bhaskara, for solving $ay^2 + s = x^2$, led to results enabling him from one integral (or rational) solution, $x = p_1$, $y = q_1$, of the more special equation $ay^2 + 1 = x^2$, to get a second solution, $p_2 = p_1^2 + aq_1^2$, $q_2 = 2p_1q_1$ and, by repetition, an infinite series of solutions of $ay^2 + 1 = x^2$.*

Perhaps less widely known is the rule given by Theon of Smyrna (about 130 A. D.) and proved by Proclus (410–485 A. D.) for finding numbers representing the sides and diagonals of squares, *i. e.* satisfying $d^2 = 2s^2$. If d_1 and s_1 is a solution, so is $d_2 = 2s_1 + d_1$, $s_2 = s_1 + d_1$. But $s_1 = 1$ and $d_1 = \sqrt{2}$ is a solution. Consequently, we get an indefinitely large number of solutions $d_2, s_2, d_3, s_3, \cdots$ Similar processes are given by Proclus for $x^2 - 2y^2 = 1$. It is believed that these problems were studied by the Pythagoreans and by peoples of still earlier times.† In these Greek and Hindu writings the recurrent process is used for finding one solution from another solution, rather than to show the generality of some particular formula or theorem. It is easy to see how some slight change in the mode of presentation or in the point of view would yield the modern mathematical induction. Thus Proclus was very close to the establishment of the formulas $d_{n+1} = 2s_n + d_n$, $s_{n+1} = s_n + d_n$, for any value of n.

All in all it seems best to restrict the term "mathematical induction" strictly to the modern definition of the term, and to ascribe it to Pascal. The recurrent modes of inference of the Hindus and Greeks are more nearly the modern process of mathematical induction than is the mode of inference used by Fermat.

FLORIAN CAJORI.

* H. Hankel, Geschichte d. Mathematik, Leipzig, 1874, p. 200.

† Fr. Hultsch, "Die Pythagoreischen Reihen, etc.," in *Bibliotheca Mathematica*, 3d ser., vol. 1, 1900, pp. 8–11; H. Konen, Geschichte d. Gleichung $t^2 - Du^2 = 1$, Leipzig, 1901, pp. 2–11.