It is used in the paper in finding general determinant values for the sums of the series

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots, \quad \frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\cdots.$$

29. A hyperorthogonal group $HO(3, p^{2k})$ which is representable as a collineation group in three variables, the coefficients being marks of the Galois field $GF(p^{2k})$, has an invariant of the form

$$x_1^{p^k+1}+x_2^{p^k+1}+x_3^{p^k+1}=0.$$

It is of order $(p^{2k}-p^k+1)(p^k+1)^2p^{3k}(p^k-1)/d$, where $d=3$ or 1, according as p^k+1 is or is not divisible by 3.

Dr. Mitchell determines the subgroups of this group for p an odd prime and k a positive integer. The least number of letters on which the group is representable as a permutation group is $p^{3k}+1$, except for $p^k=5$, in which case the least number is 50.

F. N. COLE,
Secretary.

THE PREPARATION OF COLLEGE AND UNIVERSITY INSTRUCTORS IN MATHEMATICS.

*PROVISIONAL REPORT OF THE AMERICAN SUBCOMMITTEE OF THE INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS.**

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* *International Commission on the Teaching of Mathematics.* Committee XIV —Graduate Work in Universities. Sub-committee 3 — Preparation of Instructors for Colleges and Universities. Chairman, E. B. VAN VLECK, University of Wisconsin; C. L. BOUTON, Harvard University; H. E. HAWKES, Yale University; G. D. OLDS, Amherst College; H. E SLAUGHT, University of Chicago. This report was submitted to the American Commissioners in October, 1910.

PREFACE.

In the preparation of the following report the committee has found little or no literature dealing specifically with the training of mathematical instructors for the higher institutions of learning in the United States. The subject is, indeed, one to which little explicit attention has been paid. The committee has therefore been obliged to beat out its own path. As, moreover, the usual preparation for the teaching of mathematics is inseparably connected with the history and organization of mathematical instruction in our country, it has been found desirable to treat the subject freely and discursively.

I. SURVEY OF PAST CONDITIONS.

To obtain first an adequate comprehension of the conditions prevailing today in the teaching of mathematics, it is necessary to review briefly the past. The mathematical development in the United States during the last quarter of a century has been so unexpected and sudden, and so remarkable in its proportions, that it can scarcely be termed an evolution but merits rather to be called a revolution. Thirty years ago the contrast between the mathematical and the astronomical situation in this country was most striking. In astronomy we numbered some of the world-leaders, but in mathematics there was, perhaps, not an American known abroad for investigational work. In 1876 the Johns Hopkins University was founded, and soon afterwards the English mathematician Sylvester came to this country and started a small group of investigators. Some time prior to this Benjamin Peirce had stood forth at Harvard as a unique figure, a solitary investigator in the field of mathematics. Possibly owing to his influence, there emanated from Harvard University about 1880 the policy of sending promising mathematical students abroad that they might reap the fruits of German scholarship. To a limited extent the example of Harvard was followed by other institutions. As our mathematical students returned from abroad, stamped like many another excellent product as "made in Germany," they brought with them a knowledge of the modern advance in mathematics and an appreciation of investigation. Then it was that the mathematicians of the country were wakened so as to perceive the infinitesimal development of their science in America and the boundless advance abroad. In this manner, and largely through the leadership of Johns Hopkins

and Harvard Universities, there arose a small circle of investigators, fired with a determination to bring American conditions into some sort of correspondence with conditions abroad.*

Roughly speaking, the years 1880–90 may be said to mark a new departure in the mathematical education of the United States. Prior to this time there had been a fixity of aim and a definiteness of character in the methods of instruction. The prevailing mode of instruction was the recitation system, interspersed with a few lectures. For study a text book was used in which definite lessons were doled out daily, and upon these lessons the student was compelled to study and recite. Naturally he presented his work in the best form of which he was capable so as to make the best impression possible on his teacher. The instructor in turn devoted his time to correcting the mistakes and deficiencies of the student. Thus by study of a definite text and correction by the teacher great stress was laid upon *form*, while little or no emphasis was put upon imagination and upon the student's initiative or power of selection. It was the teacher's province to select the food ; it was the student's to digest the daily allowance.

The amount of mathematics taught at this time was extremely limited. Calculus was a junior or a senior elective, and its theory was taught from the purely formal side. Spherical and practical astronomy, Salmon's Conic Sections, and at most one or two other subjects (particularly quaternions) completed the college mathematical course.

For teaching, little knowledge of mathematics was demanded beyond the subject to be taught. After graduation the ablest students were usually called upon at once to teach without further study, although after the establishment of Johns Hopkins University a steadily increasing tendency was noticeable to demand first a year or two of graduate study. But the bright student still considered it preferable to try at once his hand at teaching, looking upon this as an experience which would fit him to profit by more advanced study later. If, on the other hand, he should fail as a teacher, no time would be lost in useless study before changing his vocation.

No special preparation for teaching was then required, nor indeed was any needed ; for as a student the teacher had gone

* For further information concerning the influences which made for progress, see Professor Fiske's presidential address before the American Mathematical Society, BULLETIN, vol. 11 (1904), p. 238.

already through the machinery of the recitation mill, and this produced a certain perfection and uniformity of results. The polishing of the rough and projecting edges of the raw product through the friction of daily recitation, the demand for quick and ready expression by the student when on his feet, the insistence upon clearness, accuracy, and good English in the production of work (qualities which, it should be noticed parenthetically, were necessary to the professions of the law, ministry, teaching, into which the great majority of the students of that time passed) — all this constituted in itself an admirable preparation for teaching. Whatever may have been their deficiencies in exactness, extent, and depth of knowledge, the college mathematical professors of the time were commonly adepts in teaching the text book material, and they knew how to present it so as to make it clear to the average man's comprehension. Like produces like. In consequence the young teacher, put in without special training to "sink or swim, live or die, survive or perish" according to his ability to control and guide the boisterous elements of his classes, generally fulfilled his mission satisfactorily.

The situation can be summed up by saying that the machinery was well adapted to its end and turned out its material after the model set. On the other hand, various other qualifications of supreme importance for the teacher were too often ignored or sorely neglected; for example, initiative in study, geometric and analytic imagination, rigor, power of generalization, etc. It is, therefore, not surprising that the mathematical training of that time failed completely to produce the true scholar or the great mathematician.

II. Present Day Conditions.

The conditions today with respect to teaching are in part a survival of those of the past, and in part the result of the ingrafting of new ideals and methods. The earlier methods, in so far as they were not indigenous, were an inheritance from the English tutorial system. The changes have been due to contact with the mathematicians of the continent of Europe, and in particular with German mathematicians. As our students returned from study abroad and themselves became teachers, there sprang up a demand for up-to-date knowledge, for *thorough* scholarship and investigation. As a test and a certificate of some ability to meet these demands, the Ph.D.

degree speedily became a very common prerequisite for desirable college appointments. So rapid has been the change in recent years and so great the growth of the more prominent institutions of the country (especially the state institutions), that the supply of teachers and investigators has become inadequate and falls ever increasingly short of the demand.

While this condition exists in the more important institutions, the average small college west of the Alleghanies has been little touched by the awakening. Institutions of respectable standing with 800 or 1000 students also exist which are substantially in the condition of the college of twenty-five years ago, and this too, without reaching the degree of efficiency and excellence attained in the old time eastern college. Between the two extremes lie a multitude of colleges in which the new influences are felt in varying degree.

In the better institutions the mathematical progress has been attended by a remodelling of the curriculum. Calculus has become today primarily a sophomore study.* In the last two years of the college course a considerable variety of electives has been introduced, extending as far as elementary courses in the theory of functions of the real and the complex variable. Graduate courses cover a range of topics fully as wide as, and perhaps even wider than in Germany.

Speaking roughly, it may be said that up to the calculus the mode of teaching remains the same as of old. Here freer and more varied methods begin to prevail. While in most institutions it is still found advantageous to teach calculus by the recitation method, yet a large part of the class hour will frequently be spent by the teacher in discussion and explanation. The general view seems to be that about this time the student reaches a greater maturity of power, and the instruction he should receive requires therefore greater inspirational power and breadth of view on the part of the teacher. From this point on, the recitation recedes into the background, being superseded by the lecture or lecture-and-quiz method. The subject matter thereafter is no longer closely prescribed by precedent, and the course of instruction is laid out largely in accordance with the taste, training, and ideas of the teacher.

Not only is the calculus a boundary line between two styles

* At Harvard it was so recognized as early as 1870. This is indicative of certain tendencies toward the elevation of mathematical instruction in America before the advent of foreign influences.

of teaching, but it also marks a sharp change in the problem confronting the teacher of mathematics, a change almost as great and important as that from the gymnasium to the university in the German educational system. The alteration in the educational problem at the end of the sophomore year is so well known and recognized that it has been proposed by some to relegate eventually the work of the first two years of the college course to the high school; by others to separate the work of these two years into a junior college leading to an appropriate degree.

Without discussing the advantages and disadvantages of such a change we shall proceed to consider further the nature of the mathematical problem in our large colleges and universities before and after the calculus. It should be borne in mind that nothing can be said that will fit *all* of our universities and colleges, for their number is legion. Yet in the main the remarks below portray the situation correctly, for even in the academic department of Harvard University, where the lecture system is introduced from the beginning, it is accompanied by daily tests or problems in the elementary work.

In most of the leading institutions the mathematics of the freshman year has been made either elective or semi-elective in the college of liberal arts. Such, however, is its importance that classes in the elementary subjects of trigonometry, algebra and analytical geometry are very large and create a peculiar problem of administration. In the engineering schools the same condition holds for the calculus also. A sample illustration may be taken from the University of Wisconsin. Here in the years 1908–1909 and 1909–1910, there were each semester in the College of Letters and Science a total of from 300 to 330 students to be given instruction in freshman mathematics. For this purpose they were divided into eleven to thirteen sections. This situation necessitates a large amount of work of a somewhat routine character. In the engineering school, in which mathematics is required, practically the same condition prevails. The adoption of the lecture system for these lower classes seems unwise because of its failure to meet the individual difficulties of the student and to give him the repetition and drill necessary at that stage of his development.

Even in these more elementary classes where the recitation remains, one notices a change in the character of the problem confronting the teacher, especially in the eastern section of the

country.* College youths no longer form a carefully culled and picked intellectual body designed solely for the professions. The increasing educational development of the country has touched all classes, even the foreign immigrant, and has increasingly brought to college young men preparing for business pursuits. The country has been so inundated by foreign immigration as to become the "melting pot" of the nations, and the high school is the crucible. The pupils in the high schools, being in vast numbers the children of men and women who have learned the English language only after arriving in this country, mix foreign idioms with the grammar of the school. The classes are often very large (40 or 50 being not uncommon in a class), so that not much attention can be given to the individual; and the best students, in particular, suffer from receiving instruction appropriate to the mass. Moreover, the examination for admission to college has disappeared almost entirely and has been replaced by certification from the school.† For all these reasons the average student enters a university more poorly prepared than formerly, and brings with him less of the cultural influence that came with the select few of the earlier days. Under these conditions the freshman instructor finds it impossible to maintain in his classes the former rigorous standards of recitation, of good form and exposition in concise, lucid English. Consequently the recitation system no longer produces as satisfactory results as formerly, though it doubtless serves its purpose better than any substitute that has been suggested.

Subsequent to the calculus the number of pupils in mathematical classes is small. There remains chiefly the student preparing to teach in high schools and colleges, with a sprinkling of others who need mathematics for physics, engineering, or its other applications. The fundamental problem in teaching is henceforth to inculcate in the prospective teacher thorough scholarship and sound standards, which were necessarily somewhat neglected in the previous mass-instruction.

* In the newer sections of the country pioneer or unsettled conditions have prevailed largely from the beginning.

† Examinations are still required of everybody at Harvard, Yale, Princeton.

III. The Present Preparation for Teaching and its Deficiencies.

Owing to the changes above described the recitation system no longer affords the prospective teacher the same drill in presentation and the same feeling for form that it once did. Under the lecture system which succeeds it the emphasis is put primarily upon acquisition and upon extension of knowledge. But the mere hearing of lectures leaves the student passive and does nothing to tax actively his powers of presentation unless supplemented by other aids. (See Section IV.) Furthermore, if the teacher himself is deficient in style and is so absorbed in acquisition and investigation that he slights the matter of presentation, the student is likely instinctively to copy his example and to remain content with crude and half intelligible exposition of his ideas. Thus it has happened not infrequently that a bright student has found nothing in his university training to correct the deficiencies of early years or his natural tendencies, and has gone out to his career of teaching only to meet with an initial disastrous failure from which it was difficult and sometimes impossible to recover.

The introduction of the lecture method into our country has indeed been so rapid that an amalgamation of the old and new modes of teaching into a well proportioned, coherent and consistent educational system has been impossible. In consequence two dangers today beset the mathematician. On the one hand, there is extreme and unreasonable emphasis on the research side of activity at the expense of form and clearness and without due regard to values, any sort of investigation being prized even though of little or no intrinsic worth. This results in the elimination of any true interest in the actual teaching side of the profession. On the other hand, we see the teacher absorbed by personal interest in the student and by the pedagogical and administrative side of his work, to the neglect or exclusion of progress and true scholarship. Both deficiencies are found in our educational system. In a few institutions the first mentioned evil may be found, while the latter is conspicuous in many.

It might be thought that the difference in the educational problems before and after the calculus would result in a corresponding differentiation of teachers into two classes, the emphasis being laid in the one class upon the imparting of knowledge and in the other upon its acquisition. In France, for example, a sharp difference is made in the preparation for

teaching in the lycée and for teaching in the universities. For teaching in the lycée it is necessary to pass the Aggrégation. This is perhaps the most searching examination to be found in any educational system of Europe. It includes, in addition to a test of mathematical attainments, an examination in powers of exposition. For the latter, reliance upon general education is not sufficient, even in a nation temperamentally predisposed to considerations of form and elegance, but in anticipation provision is made for special preparation and instruction. No such distinction has ever been made in our country, although in choosing men for professorships there is clearly an increasing tendency to distinguish, according to the nature of the work demanded, between those who are primarily educators or administrators and those who are scholars and investigators.

It continues to be the custom to train both these classes of men as teachers by setting them to teach freshmen, travelling upwards by the rough and thorny road of experience.* Probably the drilling of large freshman classes affords the best training in teaching that can be given to the beginner. It possesses the further advantage that the younger instructors have more nearly the same interests as the lower-class students than do the older professors, and in consequence these students are likely to feel less hesitation in exposing their difficulties to the instructor than to the professor of greater reputation. Also the professor is saved thereby from routine teaching which after many years of repetition is liable to lose its freshness and to become somewhat stereotyped, monotonous, and irksome; and he is left free for other university problems demanding his riper experience. In many institutions, however, it remains the practice for the professor to retain one elementary class (a freshman section, or more often the introductory class in calculus) for the purpose of supervision and to keep in touch with the situation in the lower classes. The contact between professor and instructor when teaching different sections of the same class often affords the instructor most valuable training and help in teaching. It seems altogether probable that with the development of our mathematical education such work of supervisory instruction will be reserved

* The extent to which this is done is shown by the fact that out of 225 graduate students of mathematics in 24 universities, 61 were registered as assistants or instructors. In addition should be counted a certain proportion of 57 scholars and fellows.

increasingly for men of fine teaching capacity but without the taste and inventiveness for investigation.

The above system of freshman and sophomore instruction is, however, not without its serious and pernicious effects. Quite commonly fifteen or more hours of this work is loaded upon the instructor, in addition to which he has numberless quizzes and "blue books" to correct, no distinction being made in the assignment of such work between the humdrum man and the scholar of talent and promise. But in many institutions the work is brightened by the assignment of a small class in some advanced subject. In rare instances (Harvard, etc.) the proportion of "desirable electives" reaches a quarter or more of his total quota. Yet in general, one result of the system of lower-class instruction has unquestionably been the draining of intellectual life out of many able men by an excess of routine work which leaves them inadequate time to carry on their study and investigation.

The handling of the huge and ever increasing mass of students also necessitates a large amount of administrative work (not connected with any particular branch of instruction). For this the mathematical profession, in particular, has been looked to. By actual count there are to-day some twenty professors of mathematics who are either deans or administrative officers of similar rank in their respective institutions; among them are men of fine ability and scholarship whose services are thereby lost to mathematical research and progress. Even when not occupying such a position, the mathematical teacher is supposed to be competent for administrative work and is prone to receive an extra large assignment of it. But the present situation, bad as it is, is better than the past, and gives hope of further betterment in the future. Probably also the seriousness of the administrative problem has been responsible for another evil beside the absorption of so much of the scholastic ability of the country. It has helped to promote the pernicious view, still too largely "held by some people of whom one would expect better things, that a man can become a good college teacher by following about the same lines as the high school teacher, learning a very little more of his subject than he has to teach, and developing important accessory qualifications, of which the one dearest to college presidents is his capacity for administrative work."

IV. Supplements to the Lecture System.

Diverse means are employed to correct the deficiencies of the lecture and recitation systems pointed out in the last section. Among the supplements to the lecture system the setting of problems, the seminar, and the quiz should be especially mentioned.

The first of these is used relatively but little. It was introduced at Harvard about 1880. Probably the bane of the American educational system as a whole, even to the present day, is the excessively great stress laid upon the acquisition and assimilation of material. Individual initiative in treating new questions is deserving of greater recognition and stimulus. Now the aim of problem work is to encourage this. It is to be carefully distinguished from the merely illustrative example work in our lower classes and is more akin to the use of advanced problems at English universities. It prevents mere passivity on the part of the student and, if skilfully used, will stimulate the student's imagination and pass by imperceptible degrees into real original work. Its greater use and consideration in advanced mathematical instruction in our country can therefore be recommended. In exposition it is also not without very considerable value since the student, especially when stimulated to it by his professor, may find a peculiar interest and satisfaction in formulating properly his own results. But it should be accompanied by other methods to develop more actively the power of presentation and afford proper acquaintance with periodical literature and skill in its perusal.

The "quiz" is distinctly non-German in character and origin. Though native, it is somewhat akin to a species of French conférence. In the quiz a review is made of the contents of a group of lectures recently given, and this is done by discussion between student and teacher, and by sharp cross-questioning on the part of the latter. Not infrequently new material is added in this manner to the lectures already given. The advantage of the quiz is not alone to the student. The teacher is enabled thereby to keep in touch with him; and from the democratic American point of view "keeping in touch" is a cardinal element of sound teaching. The indefinite spinning of lectures by the teacher, careless whether they are being followed by the student, is foreign to our conception of education. Much could, of course, be said for the lecture system on the principle "The devil take the hindmost, O!"; for, undoubtedly, by sifting the students and casting out the weak the best talent can be most

rapidly developed. But one of the characteristic American aims is the development of an intellectual democracy rather than an intellectual aristocracy. It is greatly to be regretted that the quiz is not more frequently used, and its development more carefully studied. In the hands of the skillful teacher a quiz, say once in every four or five lectures, can be employed to instruct and aid simultaneously the weaker and the stronger pupils. Great insistence can be laid upon accuracy, clearness, conciseness, upon thorough comprehension and expression of ideas; and the slipshod work, due so often to a mere hearing of lectures, can thereby be checked. While the quiz may impede rapidity of progress in a given subject, the loss is more than compensated by the hearty cooperation and understanding which it secures between teacher and student, and by the added interest in the work.

The seminar and pro-seminar (i. e., a seminar not based upon research) are German methods adopted and adapted in the American system. In the seminar the student becomes the expositor, and oral reports are given on problems connected with his investigation or on his collateral reading. In the opinion of the committee the quiz and seminar together can be made an invaluable complement to the lecture system in the training of teachers.

Another very common aid is the mathematical club. This serves as a species of clearing house or a rallying point for the department, instructors and graduate students being brought together for discussion and report. Its advantages are very similar to those of the seminar. Here, perhaps more than anywhere else, the graduate student is anxious to show at his very best and is "tried out," to see what he can do.

In certain institutions definite courses of reading are mapped out, while in other institutions the same end is sought by collateral reading assigned in connection with regular lecture courses. The bulk of this reading is in French or German. The purpose is alike to encourage wide reading, to give facility in reading foreign treatises, and to make the prospective teacher familiar with the literature of his subject and an adept in selecting material from widely different sources.

The usefulness of the M.A. and Ph.D. theses as preparation for teaching should not be overlooked. A doctor's thesis alone has proved to be a totally inadequate preparation in written exposition. For this reason some sort of a thesis should also be required for an M.A. degree, unless some substitute is afforded,

as for example in the additional "minor Ph.D. theses" at Harvard. The experience of editors of mathematical journals reveals on the part of the average young investigator, lamentable lack of understanding as to what clear, coherent, and readable mathematical English is. Too often the editor is obliged to ask some one to assist the author by friendly criticism. Usually a remarkable improvement is noticed after the completion of three or four successive articles by the author. The responsibility for critical work of this character should not lie with the editor but with the university teachers. Nowhere can a student be taught clear exposition better than in written mathematical English, and in no department, perhaps, are the difficulties of exposition greater than in advanced mathematics. Hence in training for teaching, the use, and if possible, the repeated use of written exposition seems to the committee indispensable. The requirement of a written theme each semester in graduate courses in which no examinations are given is an expedient especially to be commended.

In many universities assistantships and fellowships are designedly used, not merely to supplement the educational force but as an aid in the training of teachers. The assistant or fellow while continuing his graduate study is asked to teach a few hours per semester; thus theory and practice are developed simultaneously. In a few institutions, notably at Chicago, there is an affiliated university high school at which selected students may also teach. In certain rare instances a peculiar cooperative scheme has been tried by the professor, who employs an advanced graduate student as his assistant in large classes. The latter is present at all class exercises, observing the methods of the teacher and taking his place in his absence. He is called upon for aid in correcting papers and to give individual assistance to the student. Thus at all stages of the class instruction he is in conference with the professor regarding the conduct of the work. This would appear to be true practical pedagogy, well worthy of further consideration and trial.

V. Teachers' Courses and Teachers' Colleges.

In recent years the pedagogic side of teaching has been recognized at the large universities and not a few colleges by the introduction of the so-called "Teachers' Courses." In a few universities courses of this character have been separated and organized into a Teachers' College, notably at Columbia Uni-

versity, the University of Chicago, the University of Cincinnati, etc. In these courses hints concerning teaching are given, attention is called to the essentials of good teaching and to the common dangers and pitfalls, and the pedagogically difficult parts of secondary mathematics are discussed. To assist the man who goes into college work it is the aim at Columbia University and the University of Chicago to "give a good knowledge of the history of mathematics, of the curricula in different countries and of the danger points which confront teachers in their early days of work." Special topics are sometimes assigned, for example, an historical study of the methods tried in the teaching of calculus.

Up to the present time the teachers' colleges have been patronized almost exclusively by elementary and secondary school teachers. They stand as a protest against the entire neglect of the pedagogical side of teaching. It is the apparent belief of these colleges that a great work can yet be done for college and university teachers as well as for teachers in the lower grades. Not only is it said sharply that "training in the art of teaching is not even a minor requirement for a degree," but it is further urged that in mathematics above all subjects it is not enough for successful teaching that a man shall know the subject he teaches; he must have power of adapting it to the average man's comprehension. For this purpose greater attention to pedagogy and to form is desirable in preparation for teaching, doubly so in a country where no fixed traditions prevail, and where the recent mathematical growth has been so rapid. Only in this way, it is asserted, can we save our schools and elementary college classes from the crude mistakes of the youthful teacher.

As yet, however, the influence of the teachers' colleges upon the preparation of college and university teachers has been felt purely as a protest. Special methods for the training of such teachers have not been there developed, and little, if anything, has been attempted in the way of actual courses for them. The whole question as to whether such courses should be given at all, and, if so, what kind of courses should be given and where — whether in such colleges or in the graduate school — remains for determination. The whole field is a virgin one and not to be developed hastily or inconsiderately. And, finally, it seems to be conceded that *dogmatic* instruction in the art and method of teaching would be injurious and disastrous in collegiate mathematics.

Little or no encroachment has, in fact, been made upon the graduate school by the professional educator. Against his view is urged the argument that there is every difference in the manner of preparing the elementary and the advanced teacher. In the elementary school the child moves in a world so different from the world of his teacher that a study of psychology and pedagogy helps to bridge the gap, but the student of eighteen to twenty is sufficiently near his teacher in maturity that such aids can be dispensed with. The deficiencies of the American college teacher are rather on the side of an accurate, clear-cut knowledge; of a failure to distinguish between the husk and the kernel and to insist without ceasing that the student too shall search for the kernel. The prevalent opinion is that definite instruction in method crushes individuality, produces regrettable uniformity, stifles life and capacity for growth. Better an ultimate success obtained after partial failure and experiment and without suppression of individuality, than a dead level success, which never rises above rule and precept. Hence in the graduate school the primary aim must ever be to build a sound mathematical foundation and to stimulate a discriminating interest and enthusiasm. The chief preparation is then to teach the student mathematics.

But after all this has been justly said, there remains with the committee the belief that the graduate instruction is arranged too exclusively with reference to research and with too little reference to the needs of the prospective teacher. A limited part of the time in the mathematical club and seminar might be devoted with profit to the more strictly professional side of the training.

Furthermore, the tendency to undue specialization in some obscure corner of mathematics — for example, of geometry or of substitution groups — should somehow be counteracted. This might perhaps be done in part by carefully planned survey courses, such for instance as Klein's masterly lectures on projective geometry or "Höhere Geometrie." In particular these survey courses could be employed to bring to the student's attention such literature and lines of study as bear directly or indirectly upon the college subjects which he is most likely to teach. And above all the need is for broad, yet incisive introductory courses in the main and vital fields of mathematics.

VI. The Amount of Graduate Study.

Broadly speaking, the minimum requirement for teaching in college and university tends to become the equivalent of the master's degree. But in the better institutions it is recognized that a year or a year and a half of graduate training is by no means enough for a permanent college or university position, and the possession of the Ph.D. degree is therefore made a sine qua non. The situation thus created is in many ways unsatisfactory in regard to both degrees.

On the one hand, the demand for graduates with the doctor's degree has had increasingly the effect of making the Ph.D. degree a sort of college teacher's certificate, and has acted in a generous measure to cheapen the degree. Too many doctors have doubtless been turned out who have no actual interest in research and seek the degree solely for its commercial value. It has been the rule rather than the exception that the holder of the Ph.D. degree fails to go on to further research.

On the other hand, the situation with respect to the master's degree is even worse. It is in no way an adequate certification for college teaching. In many institutions it has been regarded as essentially a degree for secondary teachers. In other institutions it is used with one class of students as a stepping-stone to a higher degree, with another class of students who are found not to possess the inventiveness and fertility of mind necessary for research it is both a recognition and discharge. It is indeed difficult to say for just what the M.A. degree does stand. The committee submits it as a question worthy of consideration whether the M.A. degree might not be so administered or developed as to become a desirable teacher's degree or certificate in distinction from the Ph.D. degree as a certificate of research. Thus it could be used as a recognition alike for college teachers without the doctor's degree and for secondary teachers of decidedly superior capacity. Possibly in this way the Ph.D. degree could be relieved of some of the heavy load which it is now compelled to carry.

The present sharp differentiation of college mathematicians into two classes, the holders of the master's and of the doctor's degrees is in many ways a most unfortunate one. Many men have stopped with the former who are capable of proceeding much further. The committee wishes to record its emphatic belief that every man should be encouraged to study just so far as his ability and taste may qualify him, without stopping at

the line of either degree. Many a teacher without the kind of ability necessary for research can yet be encouraged to become a thorough scholar in some definite line, instead of looking to administrative office as his career.

More systematic encouragement for study after the attainment of the doctor's degree is also a desideratum. The introduction of the German privat-docent system into this country is recognized to be impossible and totally foreign to the American spirit of financial independence after graduation. Yet some means should certainly be devised for recognizing better the more talented few and giving them opportunities for study commensurate with their ability. As before pointed out, the life and ambition of many young instructors have unquestionably been stifled by the load of fifteen or more hours of class room work, with the accompanying written exercises for correction. In the case of the very ablest young instructors some financial provision for maintenance should be made with a reduced number of hours of work. The prevalent American practice is to relieve the instructor of his heavy hours of work only when he is about forty years of age, or after he has proved himself a pronounced success by virtue of his ability or attainment. In opposition to this it cannot be too much emphasized that the age of preparation for intellectual work of high order is from twenty to thirty, or to thirty-five at the latest.

VII. The Supply and Appointment of Teachers.

The same general considerations which affect the supply and quality of teachers in other departments apply also to the teachers of mathematics. The serious feature in the situation today is a still increasing deficiency in supply relatively to the demand. This necessitates also to a considerable degree a deficiency of quality, since places must be filled by such persons as can be obtained. The deficiency in supply is primarily a result of the enormous growth of the educational system of the country, but the deficiency in quality is due also to the rapid advancement of the mathematical profession in standards and knowledge. On the absolute scale the number of good mathematical teachers in the United States has never been greater, yet never also has the shortage been greater. The expansion of the country and its increase in population has outstripped growth in education, and the outlook is that the situation will become even more acute in the immediate future. Not only must the prospective growth

of the country, especially of the middle and far west, be taken into account, but also the awakening and development of our neighbors, China and South America. This expansion at home and abroad affords a great opportunity for the engineer. It attracts and doubtless will continue to attract a large number of the ablest mathematical students away from abstract mathematics and teaching into the practice of the engineering and allied professions.

The attractions of these professions in themselves are enhanced by the remuneration. The financial rewards for the successful engineer are incomparably greater than those for the mathematician. Twenty-five years ago the standard salary for the full mathematical professor in our best colleges was $2500, the variation to either side of this figure being but slight. Today the corresponding figure is $3000, the variation ranging from $2500 in small cities to $4000 or $5000 for a university situated in a large metropolis. For assistant professors $1500 to $2500 is the current compensation. For the successful engineer the compensation must be reckoned in five place figures.

These salaries may very possibly appear adequate to the foreign professor not acquainted with American life and social conditions, but they are notoriously inadequate to the legitimate demands of life in this country. The high price of rents and cost of living, the cost of travel necessary in a country of large distances to attend conventions and to keep in touch with mathematicians at home and abroad, and many expenditures due to prevailing social conditions are recognized to press hard upon a college professor. If he has a family of three or four children, the financial pinch is likely to be felt daily and bitterly, and anxious thought must be spent upon the mode of spending one's income. Furthermore, the mathematician by the very nature of his subject is cut off from the usual modes of supplementing a professor's salary open to his colleagues, as for example by writing or by the delivery of public lectures, by legal and expert advice, by chemical or biological analysis, and so on. His only resource may be to write a text book in one or more of the four elementary subjects in which there are large college classes. Hence we see issued year after year a large number of such books, the majority of which resemble one another like peas in a pod, and contribute little or nothing to mathematical progress. Rarely does the mathematician have both time and knowledge to write — for such fame and glory

as there may be in it — the unremunerative advanced mathematical text book or treatise so much needed in the English language.

With the obvious financial consideration there acts also another powerful force pulling men away from mathematical teaching. It is felt that the engineering professions give a chance to do "something worth while." In our rapidly developing material civilization the shallow view often prevails that the man of action stands on a higher plane than the man of ideas. The latter also is invaluable for highest development.

This side-tracking of genuine mathematical talent to engineering work is most seriously felt in applied rather than in pure mathematics. Precisely here, where the American mind might be expected to scintillate with flashes of genius, there is a real poverty of talent. While we boast our late-lamented Gibbs,— born like many a precious flower to blush unseen,— it must be confessed that his career is indicative rather of the interest and talent which might exist in applied mathematics than of that which does exist. In support of this statement it suffices to point out how few university centers there are in the United States where a first class training in applied mathematics can be obtained. The diversion of students to engineering is not solely responsible for this. It is in part a consequence of past influences when mathematics was pursued in our country as a branch of logic and a purely deductive science. A tendency to the purely formal side of mathematics can be noted from the days of Benjamin Peirce on. It is therefore not much to be wondered at (though much to be regretted) that American mathematical research has inclined so much to this formal side; for example, to multiple algebras, postulational and axiomatic foundations, theory of groups, etc. Hence, there is need of great insistence today that applied mathematics shall be more abundantly pursued and for a longer period; that men of sound mathematical training shall be brought into touch with vital physical problems.

The current method of filling vacant positions is the Socratic one of question and answer. The professor in charge of the department or some one of its divisions first makes necessary inquiries, writing particularly to the mathematical centers where the graduate students are most likely to congregate. When practicable, he supplements the information by personally meeting the candidate under most favorable consideration and thus

gains first-hand impressions. If the position to be filled is a major and not a minor one, the candidate is frequently invited to come to the university for a visit, to inspect and to be inspected. The considerations governing a choice of candidates are discussed with the president, or president and dean, and usually the choice of the professor is accepted as a result of the conference. In some cases the selection must be approved by an academic council consisting of some or all of the full professors. Under normal circumstances the subsequent ratification by the board of regents or trustees is a matter of form, since the selection of instructors is properly deemed a prerogative of the president and board of instruction.

When the head of a department is to be chosen, a common practice is to appoint a committee of from three to five members, inclusive of the president or dean and containing representatives of other departments than the one concerned. This committee then seeks for information in all possible ways.

Applications for a vacant position are often forwarded by those who would be glad to secure the appointment. To a large extent these applications are waste paper. Wider publication of vacancies has been advocated by some; but it is the almost universal experience of those who have the appointments to make that a too wide advertisement results in a bothersome and useless hoard of applications from men, the number of whose submitted testimonials varies inversely as the square of their fitness. Information privately solicited is trusted more than the public testimonial. In the case of well-known men a hint of their availability is as effective as any formal application, but in the case of minor appointments information supplied by a really worthy applicant is sometimes a most important factor in the decision for his appointment.

To fill vacancies in small or less important colleges, the teacher's agencies are often consulted, as by many normal and secondary schools. In such cases the standing of the college and the salary are not sufficient to attract the better trained men secured by the methods just described. These institutions rely particularly upon their recent and brightest graduates to fill the vacancies. To a great extent the practice of "inbreeding" by selection from the alumni of the institution has been current in our country, and this has been true even of some of the largest and most important of our institutions. When carried to excess, the practice cramps and narrows the development of

the college or university. It is increasingly felt that the best results are obtained by generous infusion of instructors having other ideas and training than those characteristic of the institution itself.

It cannot be said that mathematical proficiency always dictates the appointments. Not infrequently there are better men available. Yet this is no doubt less often the case than is realized, since frequently appointments which are severely criticised have been made only after better men have been sounded and have declined to be considered for the position. Other qualifications than mathematical proficiency must receive consideration, such as enthusiasm and the ability to stir mathematical interest in students, personality and the genius to gather and keep coherent a group of promising instructors, ability to develop a department so as to make it useful and respected in the institution, and so on. Qualities such as these will doubtless weigh more in the future than in the past, and administrative ability will be less considered owing to the increasing separation of business work. Doubtless in the past many sins of bad mathematical appointments must be laid to considerations of administrative ability.

Two qualities may be noticed as especially needed by the American teacher. The first is a broad, liberal culture. The pursuit of mathematics in itself is doubtless a narrowing one. There is perhaps no science of which the development has been carried so far, which requires greater concentration and will power, and which by the abstract height of the qualities required tends more to separate one from daily life. A wide liberal culture therefore is eminently desirable for the establishment of that attractive personality so necessary for the best success in teaching.

The second quality, moral fibre and force, is demanded largely in all institutions but especially in the small American college where teacher and student come into particularly close personal relations. No just report concerning the selection and distribution of mathematical teachers can be given without some reference to this fact. The denominational origin of most of the eastern and older colleges accounts for the emphasis laid upon the possession of these qualities. While denominational lines have broken down and tended to disappear, and accordingly religious distinctions have been forced increasingly into the background in college and university education, the moral

ideal nevertheless survives, receiving perhaps a greater consideration than in any other country except England. In the junior college — to designate thus the first two years of the college course — the student is still young and immature and in a formative period. Here the public claims that the student has as much a right to the care, time, and thought of the teacher as has investigation. It insists, even in the large state universities, that the child shall be cheated neither out of its intellectual nor out of its moral birthright. The mathematical teacher must with rare exceptions begin his career either in the small college or in the junior college of the large university. He consequently must have the problems of the junior college at heart, and not neglect his students for his own investigation. Not a few talented men fail altogether because of such neglect or lack of interest and thereby close their mathematical career.

In the graduate school the considerations of the last section fall mostly into abeyance since the student is now sufficiently mature to criticise rather than copy the deficiencies of his instructor. Intellectual leadership and vitality are here the crucial question. With increasing separation between the junior and senior colleges there comes an increasing difference in the kinds of teachers required.

The question, "From what class of society are our mathematical teachers drawn," is a complicated one, not to be answered by statistics. Have they come up through poverty and hardship, liking mathematics for its difficulties? Do they come from the great middle classes, or from well-to-do cultured homes? Undoubtedly all classes contribute, but the chief source is probably the second, especially that section of the middle classes which is more poor than rich and the section which lies at the fringe of the cultured classes. To men from these homes the teacher's profession offers improved social conditions and increased opportunity. Its financial deprivations are borne with sufficient philosophy and complaisance, for the love of intellectual pursuits is the life of the teacher.

The large number of mathematical teachers coming from the very small college, where the curriculum is cramped and where mathematics is one of its chief components, should be especially noted. Probably these colleges furnish an abnormally large percentage of the mathematical students for the graduate school.

VIII. Concluding Remarks.

In conclusion, the committee abstains from prophesy or the formulation of any definite and ideal program for the training of the advanced mathematical teacher. The situation regarding such training is still chaotic, and there is no method yet apparent of evolving order. The aim of this report has been, rather, to point out the causes, results, and deficiencies of the present method of training (or lack of method), and it will be enough if the committee shall have succeeded in arousing interest and stimulating thought upon the subject.

Certain general points may, however, be noted by way of summary or emphasis.

The introductory courses in calculus, as already indicated, forms really the beginning of the training today. Prior to this the study of mathematics is pursued chiefly as a practical and disciplinary training for the general college body, while thereafter it is pursued by the mathematician for its own sake.

The instruction of the mathematician and physicist for a longer time together in identical courses would undoubtedly have its advantages. Certainly the greater intermingling of pure and applied mathematics is desirable. The extent to which they have been separated is astonishing. The pure mathematician is prone to be either helpless or ignorant before the applications, while the applied mathematician has rarely a good grip on modern mathematical principles. Not in a divorce from related subjects, but rather from the inspiration of contact with them is the continued life and progress of that science to be sought which is the most perfect and coherent development of human thought.

The committee expresses the further wish that the early training of the mathematician on the purely mathematical side shall not be too narrow. The four great fields, analysis, mechanics, geometry, algebra (number fields, etc.) should early be represented strongly in their *main* essentials and characteristics.

The matter of training in clear, coherent, and interesting presentation both oral and written should not be overlooked, as is sometimes done by the teacher, but he should be willing to give his time generously to their inculcation. A more abundant practice therein should be afforded for the future teacher and investigator.

Probably greater cooperation between the experienced teacher and the beginner regarding the conduct of elementary college

instruction is needed. This might be gained by more abundant discussion of the problems involved, by helpful suggestion and mutual visitation of classes, and occasionally even by the joint conduct of a class.

Finally, the amount of routine instruction, at least for the most promising of our younger college and university investigators, should be so regulated as to further rather than suppress their individual development. On the other hand, the man with a gift for teaching, though not for investigation, should be encouraged to obtain a broad and comprehensive knowledge of mathematics, and, obtaining this, should not be cashiered for failure to "produce" nor be spoiled as a teacher by being trimmed into a very mediocre investigator. The difference between the two kinds of gifts — the power to teach and the ability to extend the frontiers of our science — should be more clearly recognized. It is for a free and not a standardized development that we plead; and, above all, for greater freedom and leisure for the most able of our younger instructors that they may achieve the best that is in them.

VECTOR ANALYSIS.

Vector Analysis. By JOSEPH G. COFFIN. New York, John Wiley and Sons, 1909. xvii + 248 pp.

Einführung in die Vektoranalysis mit Anwendung auf die mathematische Physik. Zweite Auflage von Prof. Dr. RICHARD GANS. Leipzig, Teubner, 1909. x + 125 pp.

Die Vektoranalysis und ihre Anwendung in der theoretischen Physik. Von Dr. W. v. IGNATOWSKY. Theil I: *Die Vektoranalysis*, 1909, vi + 112 pp. Theil II: *Anwendung der Vektoranalysis in der theoretischen Physik*, 1910, iv + 123 pp. Leipzig, Teubner.

A GLANCE at books of the above type calls attention to the unsatisfactory position that vector analysis still occupies in our courses of study. We find here discussions of the most elementary kind mixed with other discussions requiring a considerable knowledge of mathematics and physics. To the student unacquainted with the branches of science involved, vector analysis is hardly intelligible. To the student acquainted with them, it seems superfluous as leading merely to results already