THE EIGHTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The eighteenth annual meeting of the Society was held in New York City on Wednesday and Thursday, December 27–28, 1911, the programme occupying two sessions on each day. The total attendance numbered about seventy-five, including the following sixty-four members:

Dr. C. S. Atchison, Mr. H. Bateman, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor B. H. Camp, Professor W. M. Carruth, Dr. A. S. Chessin, Professor C. W. Cobb, Dr. Emily Coddington, Professor F. N. Cole, Professor J. L. Coolidge, Dr. L. L. Dines, Mr. E. P. R. Duval, Professor L. P. Eisenhart, Professor T. C. Esty, Professor F. C. Ferry, Professor J. C. Fields, President H. B. Fine, Professor W. B. Fite, Mr. Meyer Gaba, Professor A. S. Gale, Professor O. E. Glenn, Mr. G. H. Graves, Professor C. C. Grove, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Professor L. A. Howland, Professor L. S. Hulburt, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Professor W. R. Longley, Professor C. R. MacInnes, Professor J. H. Maclagan-Wedderburn, Professor James Maclay, Professor G. A. Miller, Professor C. L. E. Moore, Professor Richard Morris, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Mr. L. P. Siceloff, Mr. C. G. Simpson, Professor P. F. Smith, Mr. W. M. Smith, Professor H. W. Tyler, Professor Anna L. Van Benschoten, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. E. E. Whitford, Dr. W. A. Wilson, Professor E. B. Wilson, Professor F. S. Woods, Professor J. W. Young.

President Fine occupied the chair, being relieved at the Wednesday afternoon session by Ex-President White. The Council announced the election of the following new members of the Society: Professor Ida Barney, Rollins College; Professor Louis Brand, University of Cincinnati; Professor C. W. Cobb, Amherst College; Professor J. C. Fitterer, University of Wyoming; Mr. G. H. Graves, Columbia University; Dr.

Solomon Lefschetz, University of Nebraska; Mr. G. H. Light, Purdue University; Mr. E. S. Palmer, Rutgers College; Professor E. R. Smith, Pennsylvania State College. Eight applications for membership in the Society were received.

The annual dinner on Wednesday evening was a very pleasant occasion, forty-two members being present. These informal gatherings have long been recognized as one of the most attractive features of the meetings.

The reports of the Treasurer, Auditing Committee, and Librarian have appeared in the Annual Register. The membership of the Society has risen to 668, including 62 life members. The total attendance of members at meetings during the year just passed was 423; the number of papers read was 217. At the annual election 197 votes were cast. The Society's library now contains 3,871 volumes, beside some 500 unbound dissertations. The Treasurer's report shows a balance of $8,723.89, but the increase is offset by outstanding bills amounting to about $550. The income from sales of the Society's publications during the year was $1,513.66. The life membership fund now amounts to $4,137.17.

With the great growth of the Society's activities it will soon become necessary to make some provision for clerical services, the capacity of the several offices being already overstrained in this direction. The Society has outgrown the stage in which its officers could attend personally to all the details of administration.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

Vice-Presidents, Professor H. F. BLICHFELDT,
Professor HENRY TABER.

Secretary, Professor F. N. COLE.

Treasurer, Professor J. H. TANNER.

Librarian, Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor E. W. BROWN,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1914.

Professor A. B. COBLE, Professor OSWALD VEBLEN,
Professor E. W. DAVIS, Professor E. B. WILSON.

The following papers were read at this meeting:

(1) Mr. W. M. Smith: "A characterization of isogonal and equitangential trajectories."

(2) Professor C. L. E. Moore: "Surfaces in hyperspace which have a tangent line with three-point contact passing through each point."

(3) Dr. J. E. Rowe: "How to find a set of invariants for any rational curve of odd order."

(4) Dr. J. E. Rowe: "A covariant point of the R^4, and a special canonical form."

(5) Dr. R. L. Moore: "On sufficient conditions that an integral equation of the second kind shall have a continuous solution."

(6) Professor E. B. Wilson: "Some mathematical aspects of relativity."

(7) Professor Edward Kasner: "Families of surfaces related to an arbitrary deformation of space."

(8) Dr. H. B. Phillips and Professor C. L. E. Moore: "Algebra of plane projective geometry."

(9) Professor Anna L. Van Benschoten: "Products of quadric inversions and linear transformations in space" (preliminary report).

(10) Professor Arthur Ranum: "N-dimensional spreads generated by ∞^1 flats."

(11) Professor O. E. Glenn: "Generalizations of a theorem on reducible quantics, due to Eisenstein."

(12) Professor F. R. Sharpe: "Finite groups of birational transformations in the plane."

(13) Professor John Eiesland: "On a flat spread-sphere geometry in an odd dimensional n-space."

(14) Professor C. N. Moore: "The summability of the double Fourier series, with applications."

(15) Professor S. E. Slocum: "A general formula for torsional deflection."

(16) Professor G. A. Miller: "Groups which contain a given number of operators whose orders are powers of the same prime."

(17) Professor R. G. D. Richardson: "Theorems of oscillation for three self-adjoint linear differential equations of the second order with three parameters."

(18) Professor L. A. Howland: "Points of undulation of algebraic plane curves."

(19) Professor C. N. HASKINS: "Note on certain selective integrals."

(20) Professor L. P. EISENHART: "Ruled surfaces with isotropic generators."

(21) Professor J. W. YOUNG: "On algebras defined by groups of transformations."

(22) Professor J. W. YOUNG: "A generalization to 3-space and to n-space of the inversion geometry in a plane" (preliminary communication).

(23) Dr. L. L. SILVERMAN: "On absolute or unconditional summability."

(24) Dr. L. L. SILVERMAN: "Tests for Cesàro summability."

(25) Professor W. B. FITE: "Note on a collineation group in n variables."

(26) Professor L. P. EISENHART: "Congruences of minimal lines in a four-space."

In the absence of the authors the papers of Dr. Moore, Dr. Sharpe, Professor Eiesland, Professor C. N. Moore, Professor Slocum, and Dr. Silverman were read by title. Abstracts of the papers follow below.

1. Cesàro has shown that in any complete isogonal system the locus of the centers of curvature of the single infinity of trajectories passing through the same point is a straight line. Similarly, Scheffers has proved that in any complete equitangential system those trajectories tangent to the same straight line have their centers of curvature upon a straight line. The former is here referred to as the Cesàro property and the latter as the Scheffers property.

In any doubly infinite system of plane curves any one of the single infinity of trajectories through a point of the plane has a certain curvature γ. Let N denote the rate of variation of γ along the normal, the direction of the consecutive trajectory remaining the same, and let γ_1, N_1 denote the corresponding quantities for that trajectory which is orthogonal to the first. Mr. Smith shows that a doubly infinite system of plane curves whose single infinity of trajectories through any point possess the property expressed by

$$N + N_1 = \gamma^2 + \gamma_1^2$$

together with the Cesàro property is a complete isogonal system.

If it possesses this property together with the Scheffers property, it is a complete equitangential system.

2. In this paper Professor Moore shows that the coordinates of a surface in hyperspace, having through each point a tangent line with three-point contact, must satisfy a partial differential equation of parabolic type. If the coordinates satisfy such an equation, the surface will be of the kind mentioned. Similar results are obtained for other spreads in hyperspace.

3. An algebraically complete system of invariants of the rational plane quartic has been found by Dr. Rowe in a previous paper presented to the Society September 6, 1910. (See *Transactions*, July, 1911.) This enables us to find a set of invariants for any rational plane curve of odd order, which is called R^{2n+1}. Every R^{2n+1} has a covariant rational line quartic ρ^4, whose parametric equations may be written down, and from these the four fundamental invariants for each ρ^4 may be calculated. This work has been carried out in detail for the R^5.

4. In Dr. Rowe's second paper a covariant point of the R^4, not on the R^4, is found which possesses the following properties: (1) tangents to R^4 from this point are apolar to the six flex parameters and to the q's each taken three times; (2) the line cutting out the q's on R^4 and the line cutting them out of Stahl's conic N meet in this point; (3) the flex lines of two cubic osculants whose parameters are given by a binary quadratic meet in this point. The coordinates of the point are of degree four in the coefficients of the parametric expressions of the R^4.

In the second part of the paper a special canonical form of the R^4 is discussed which enables us to express in terms of the four fundamental invariants conditions on the R^4 which are rather difficult to attack directly. For instance, the condition that two of a set of covariant parameters, cut out of the R^4 by a covariant curve, form a node is easily handled by means of this canonical form. The flecnode and tacnode conditions occur as special cases of this.

5. Making use of a theorem on which he reported at the April, 1911, meeting of the Society (cf. BULLETIN, volume 17, No. 10, page 513), Dr. Moore proposes to show that

if $f(x)$ is continuous throughout the interval

$$I: \quad a \leqq x \leqq b$$

and $k(x, y)$ is a limited function of x and y in the square

$$S: \quad a \leqq x \leqq b, \quad a \leqq y \leqq b,$$

and if the Fredholm determinant D is not 0, and finally if on each parallel to the axis of x or of y the points of discontinuity, if any, of $k(x, y)$ which lie in S form a point set of linear measure 0, then the integral equation of the second kind

$$u(x) = f(x) + \int_a^b k(x, y)u(y)dy$$

has a solution $u(x)$ which is continuous throughout I.

6. Professor Wilson comments on some points of a mathematical nature in a memoir which he and Professor G. N. Lewis, of the Massachusetts Institute of Technology, are writing on the principle of relativity. The occurrence of a parabolic non-euclidean geometry and of a corresponding non-circular vector analysis, and the possibility of the use of potentials in four dimensions are the two chief subjects treated. The former was in part set forth and in part implied in Minkowski's work; the latter has been suggested by Sommerfeld.

7. Professor Kasner discusses the existence and determination of surfaces which are affected conformally, or more specially isometrically, by a given deformation of space. Unless the transformation of space is conformal there cannot be more than six simply infinite families of surfaces affected conformally; and unless the deformation is a displacement there cannot be more than two families affected isometrically.

8. From the assumption that the sum of two points obeys the ordinary laws of algebraic addition, Dr. Phillips and Professor Moore show that there are just two kinds of addition possible, depending upon the interpretation given to a number times a point. If λA is coincident in position with A, then the sum of two points is the harmonic of a fixed line f with respect to the two points. If λA differs in position from A, then the

sum of two points becomes similar to the ordinary vector sum. This second kind of addition leads to a sort of vector analysis in which vector points appear as well as vector lines. Both additions have two fundamental or exceptional elements consisting of a line and a point.

Distance is defined as a scalar quantity uniquely determined by two points and such that distances AB along a line are proportional to the corresponding vectors AB. On these assumptions is built a theory of distance. As a consequence of the definition, to each line of the plane is assigned a definite positive direction. Angle is taken as the exact dual of distance. In the geometry in which distance and angle are defined as above, the locus of points equidistant from A is a line. Also it is found that any three parts of a triangle determine it.

It is assumed for the area of a triangle that, holding the vertex fixed and moving the base of constant length along a line, it will remain constant. It is then found that the area must be a multiple of the sum of the sides.

The product AB of two unit points is defined as the segment connecting the two points. The product of three unit points is defined as the area of the triangle having the three points as vertices. Similar definitions are given for the products of lines.

9. In this report Miss Van Benschoten presents some types of (2, 2), (2, 3), (3, 3), and (3, 4) birational transformations which are obtained as products of quadric inversions and linear transformations in space of three dimensions. It is proved that the (3, 3) transformation in which for each variable its reciprocal is substituted can be resolved in $\frac{1}{2}n(n+1)$ different ways into the product of $n-1$ inversions and a linear transformation in space of n dimensions.

10. This paper is a continuation of the one read by Professor Ranum at the summer meeting of the Society. Taking the ith tangent spread of the jth focal spread of the kth tangent spread of a given spread S, and letting i, j, k vary, we obtain a finite system of spreads determined by S. A number of properties of this system are found. Among the theorems proved is the following:

If S and its 1st, $\cdots$, $(i-1)$-th tangent spreads are all of rank r and its ith tangent spread S_1 of rank $s_1 < r$, while its

1st focal spread S_2 is of rank $s_2 < r$ ($s_1 + s_2 \geqq r$), and if either T_1, the ($i + 1$)-th focal spread of S_1, or T_2, the ($i + 1$)-th tangent spread of S_2, is of rank s_1+s_2-r, then the other is also of the same rank and T_1 is the ($i + 1$)-th focal spread of T_2. This remarkable case first occurs in 9-dimensional space.

11. Eisenstein's theorem may be stated as follows: A necessary condition that a binary form α_x^m with integral coefficients a_0, a_1, $\cdots$, a_m, all of whose coefficients except a_0 are divisible by a prime number q, should be reducible in the absolute field is that

$$a_m \equiv 0 \pmod{q^2}. \tag{1}$$

Professor Glenn proves corresponding theorems for p-ary forms. He finds that the criteria imposed by these theorems become sharper as p increases, due to the fact that the number of conditions corresponding to (1) is, in the general case, an increasing function of p. A class of theorems on the existence of a factor of a p-ary form whose order lies within limits is established. A new method of resolving a reducible p-ary form is developed. It is proved for a linearly factorable p-ary form that, if a certain related binary form is linearly factorable in a given domain, the p-ary form is reducible in the same domain, though not necessarily completely (linearly) reducible unless the factors of the binary form are all simple. Necessary arithmetical conditions that the p-ary form be completely reducible in that domain are then developed. These take the form of higher order congruences in one unknown. The last section of the paper is devoted to the theory of these congruences.

12. Noether* has shown that a single plane can be depicted upon a double plane so that the double curve is a sextic with two coincident triple points. S. Kantor† has proved that the finite group of birational transformations having 8 fundamental points in the single plane is isomorphic with the group of linear transformations which leave the sextic in the double plane invariant. Wiman‡ used the Grassmann depiction of the single plane upon a cubic surface and thence upon the double plane.

* *Erlangen Berichte*, 1878.
† *Acta Math.*, vol. 19.
‡ *Math. Annalen*, vol. 48.

He also completely enumerated the linear transformations of the sextic. In Professor Sharpe's paper a method is developed for determining the transformation of the single plane which corresponds to a given linear transformation of the double plane. It is known that there are 120 tritangent conics to the sextic which are also tangent to the sextic at the triple point. It is shown that each conic leads to a partial determination of the cubic surface and that the cubic surface can then be completely determined in 28 ways. In a special case the 120 conics and the transformations of the single plane are completely determined.

13. Professor Eiesland's paper is in abstract as follows: A generalization of Lie's sphere geometry becomes possible if, instead of using the straight line as space element, we introduce a flat spread (hyperplane) of $\frac{1}{2}(n-1)$ dimensions. In fact, since to spheres that touch must correspond intersecting flats, the condition for such intersection must be unique; moreover, $\frac{1}{2}(n-1)$ must be an integer, hence, a flat spread-sphere geometry is possible only in a space of an odd number of dimensions. It is also necessary that the number of essential parameters of the flat shall be equal to $n+1$, which is the number of parameters of the sphere.

It has been shown that for an odd space Lie's contact transformation can be generalized and that this new transformation converts the ∞^n spheres in S_{n-1} (n even)

$$\sum_1^{n-1}(x_i-\xi_i)^2+\xi_n^2=0$$

into the ∞^n flat spreads

$$x_i=ay_i+b_i, \quad z=\sum_1^{n-2}c_iy_i+d \qquad (i=1,2,\cdots,\tfrac{1}{2}(n-2))$$

in a space

$$\bar{S}_{n-1}(x_1, y_1, x_2, y_2, \cdots, x_{\frac{1}{2}(n-2)}, y_{\frac{1}{2}(n-2)})$$

with the following relation between the ξ_i's and the parameters a, b_i, c_i, d:

$$\begin{aligned}\xi_1=-\tfrac{1}{2}(b_1+c_1), \quad \xi_2=\tfrac{1}{2}i(b_1-c_1), \cdots,\\ \xi_{n-2}=\tfrac{1}{2}i(b_{\frac{1}{2}(n-2)}-c_{\frac{1}{2}(n-2)}),\\ \xi_{n-1}=\tfrac{1}{2}(a-d), \quad \xi_n=-\tfrac{1}{2}i(a+d).\end{aligned} \tag{1}$$

The condition that two flats shall intersect is

$$(\Sigma c_i' - c_i)(b_i' - b_i) + (d' - d)(a - a') = 0,$$

which, on introducing the parameters ξ_i from (1), becomes the condition that two spheres shall touch.

The transformation in question is

$$(2)\quad \begin{aligned} X_1 &= \frac{(y_1 - p_1)(\Sigma y_i q_j - z) - (1 - \Sigma p_j y_i)(x_1 + q_1)}{2(1 - \Sigma p_j y_i)}, \\ X_2 &= i\,\frac{(y_1 + p_1)(z - \Sigma y_j q_i) + (1 - \Sigma p_i y_j)(x_1 - q_1)}{2(1 - \Sigma p_j y_i)}, \\ &\cdots\cdots\cdots\cdots \\ X_{n-1} &= \frac{\Sigma y_j q_i - z}{1 - \Sigma p_j y_i}, \\ P_1 &= \frac{y_1 - p_1}{1 + \Sigma p_j y_i}, \quad P_2 = \frac{-i(y_1 + p_1)}{1 + \Sigma p_j y_i}, \cdots, \\ P_{n-2} &= \frac{-i(y_{\frac{1}{2}(n-2)} + p_{\frac{1}{2}(n-2)})}{1 + \Sigma p_j y_i}. \end{aligned}$$

Furthermore, this transformation converts the lines of curvature on an $n - 2$ dimensional spread in S_{n-1} into curves on the corresponding surface which are the analogues of asymptotic curves on a surface in 3-space. These lines have therefore been called asymptotic lines. Through any point on the surface pass $n - 2$ such lines and no more.

Finally *all* the contact transformations of this kind are considered. It is found that they are obtained by superposing the transformation (2) on the transformations of the group of contact transformations in S_{n-1} which change spheres into spheres, and also on the duality transformation. To the before mentioned group in S_{n-1} corresponds in $\bar{S}_{n-1}$ a group which leaves invariant the ∞^n flats, and transforms asymptotic lines into asymptotic line. This group is a subgroup of a group which is similar to the group of contact transformations discussed by Lie in the second volume of his large work on transformation group, chapter 25, where he shows that the transformations of the group leave invariant a hypersurface

$$z = a + \Sigma b_i x_i + \sum_{ik}^{1, \dots, n-1} c_{ik} x_i x_k.$$

14. Fejér's idea of studying the summability of the Fourier development of an arbitrary function can be extended to the development in a double Fourier series of a function of two variables. In Professor Moore's paper this extension is carried out, and it is shown that the Fourier development of a function of two variables, $f(x, y)$, continuous in the region $(-\pi \leqq x \leqq \pi, -\pi \leqq y \leqq \pi)$, except for a finite number of lines of discontinuity consisting of parallels to the axes or of curves that cut such parallels in a finite number of points, will be summable to the value of the function at every point in the interior of the region at which the function is continuous, provided the discontinuities of the function are of such a nature that the double integrals in the coefficients of the Fourier's series converge. Along the lines of discontinuity the series will be summable at points where the function approaches a definite value as we approach the point from either side of the line, and its value will be half way between the limiting values provided the line of discontinuity has a tangent at the point. The definition of summability adopted is analogous to that adopted by Cesàro for ordinary series. The detailed formulas for positive integral orders of summability have been given in the abstract of a paper previously presented to the Society*; the present paper considers only summability of order one.

Furthermore, the present paper shows how the results stated above, together with the results of the previous paper just referred to, can be applied to the solution of certain problems in the flow of heat.

15. In a previous article (*Journal of the Franklin Institute*, April, 1911) Professor Slocum has deduced a general formula for shearing deflection, and has also given a new and simplified proof of the Fraenkel formula for flexural deflection. In the present paper this work on deflection is completed by the derivation of a general formula for torsional deflection.

---

* BULLETIN, vol. 17 (1911), p. 530. There is an error in condition (*b*) of the theorem stated in this abstract. The two first equations of this condition should be replaced by the equations

$$\lim_{m=\infty} m^k \sum_{j=\mu_m}^{j=\nu_m} j^k \,|\, f_{m,j}(\alpha, \beta) \,| = 0, \quad \lim_{n=\infty} n^k \sum_{i=\mu_n}^{i=\nu_n} i^k \,|\, f_{i,n}(\alpha, \beta) \,| = 0,$$

where μ_m and ν_m vary with m, and μ_n and ν_n vary with n in any manner whatsoever.

Let M = torsional moment, or torque, acting in a plane perpendicular to the axis of the piece,
ϑ = torsional deflection, or angle of twist, produced by M,
q = shearing stress per unit area of cross section,
q_x, q_y = rectangular components of q,
φ = angular, or shearing, distortion corresponding to the stress q,
G = modulus of rigidity, or shear modulus,
W = internal work of deformation,
A = area of cross section,
l = length of member.

The internal work of deformation is first found in the form

$$W = \frac{1}{2G}\int_0^l \int_0^A q^2 dA dx.$$

For a prismatic piece of constant cross section this becomes

$$W = \frac{l}{2G}\int_0^A q^2 dA.$$

The following theorem is then proved:

If ϑ denotes the torsional deflection produced by a torque M, and W the total work of deformation arising from a torque $M + K$, then

$$\vartheta = \left[\frac{dW}{dK}\right]_{k=0}.$$

Inserting in this expression the value previously found for W in terms of the shearing stress q, the required formula for torsional deflection is obtained, namely

$$\vartheta = \frac{l}{GM}\int_0^A q^2 dA.$$

To illustrate the application of this formula, it is then used for calculating the torsional deflection of prismatic pieces the cross sections of which are respectively circular, elliptical, triangular, rectangular, and square.

16. If p^m is the highest power of the prime number p which divides the order of a group G, the number of the operators of G whose orders are powers of p is a multiple of p^m, according

to a theorem due to Frobenius. It is, however, not possible to construct a group in which this number is an arbitrary multiple of p^m. In fact, it is at least p^{m+1} whenever it exceeds p^m, and it is at least $2p^{m+1} - p^m$ whenever it exceeds p^{m+1}.

Professor Miller proves, as a special case of a general theorem, that a necessary and sufficient condition that any group contains exactly $p + 1$ Sylow subgroups of order p^m is that it contains exactly p^{m+1} operators whose orders are powers of p. He also established the following theorems: If every possible pair of Sylow subgroups of order p^m in a given group has p^{m-1} common operators, then all of these subgroups have the *same* p^{m-1} common operators. If any group contains less than $(p + 1)^2$ Sylow subgroups of order p^m, then all the operators which are common to some pair of these subgroups must be common to all of them. If G contains exactly $2p^{m+1}p^m$ operators whose orders are powers of p, it contains also exactly $2p + 1$ subgroups of order p^m and all of these involve the same p^{m-1} operators.

17. At the April meeting of the Society, Professor Richardson gave necessary conditions and sufficient conditions for the existence of solutions of the Klein oscillation problem for two equations with two parameters. In the present paper he discusses the problem for three equations with three parameters,

$$[p_i(x_i)u_i'(x_i)]' + q_i(x_i)u_i(x_i) + [\lambda A_{i1}(x_i) + \mu A_{i2}(x_i)$$
$$+ \nu A_{i3}(x_i)]u_i(x_i) = 0, \quad p_i(x_i) > 0 \quad (i = 1, 2, 3).$$

In order that there exist solutions $u_1(x_1)$, $u_2(x_2)$, $u_3(x_3)$ which satisfy the boundary conditions

$$u_1(a_1) = u_1(b_1) = 0, \quad u_2(a_2) = u_2(b_2) = 0,$$
$$u_3(a_3) = u_3(b_3) = 0,$$

and oscillate l, m, n times inspectively, it is necessary that parameter values $\lambda^{(1)}$, $\mu^{(1)}$, $\nu^{(1)}$; $\lambda^{(2)}$, $\mu^{(2)}$, $\nu^{(2)}$; $\lambda^{(3)}$, $\mu^{(3)}$, $\nu^{(3)}$ exist such that $\lambda^{(i)}A_{i1} + \mu^{(i)}A_{i2} + \nu^{(i)}A_{i3} \leqq 0$ for at least one value of the variable x_i in the interval a_ib_i and that $\lambda^{(i)}A_{j1} + \mu^{(i)}A_{j2} + \nu^{(i)}A_{j3} \geqq 0$, $\lambda^{(i)}A_{k1} + \mu^{(i)}A_{k2} + \nu^{(i)}A_{k3} = 0$ for all values of the variables x_j, x_k in the intervals a_jb_j, a_kb_k respectively ($i = 1, 2, 3$; $j = 1, 2, 3$; $k = 1, 2, 3$; $i \neq j \neq k$). The sufficient conditions for $l \geqq l_1$, $n \geqq n_1$, $m \geqq m_1$ ($l_1 = 0$ when $q = 0$) are the same when the sign $\leqq$ is replaced by the sign $<$.

Another sufficient condition (in this case for all values of l, m, n) is that the determinant of the A's be different from zero for all values of the variables. The solution is then unique.

This paper is to appear in the *Mathematische Annalen.*

18. Points of undulation or, in general, points of an algebraic plane curve at which the tangent has contact of order higher than 2 receive little attention in elementary discussions of the theory of such curves. Or, in other words, although the behavior of the Hessian in multiple points of the curve is treated as fully as may be by elementary methods, the behavior of the curve in multiple points of the Hessian, the consequence of contact between curve and Hessian, or, more generally, of multiplicity of intersection of curve and Hessian greater than $3k(k-1)$ in an ordinary k-fold point are not treated at all.

The object of Professor Howland's paper is to show that these questions can be discussed with the same degree of completeness by elementary methods as can the theory of ordinary inflexions. Among other results the formula $\Sigma(\sigma - 1) = 3n(n-2)$, which applies to curves without multiple points, is extended to curves having ordinary multiple points, in the form $\Sigma(\sigma - 1) = 3n(n-1) - \Sigma 3k(k-1)$, where σ is the order of contact of curve and tangent at each point and Σ is extended to all points of the curve.

19. Landau* and Lebesgue† have recently discussed certain formulas by means of which a given continuous function can be expressed to a preassigned degree of approximation by means of an analytic function. The latter, in particular, has given an exceedingly general theorem by means of which such approximate representations can be constructed. The note of Professor Haskins discusses a similar theorem, somewhat less general than that of Lebesgue, which, however, throws some light on the mechanism of the process of representation of a function by means of a definite integral. The theorem is: If in the interval $a \leqq x \leqq b$ $f(x)$ is continuous,

* *Rendiconti del Circolo Matematico di Palermo*, vol. 25 (1908), p. 337.

† Ibid., vol. 26 (1908), p. 325.

and $\varphi(x)$ is continuous and positive and attains its maximum value M only once in the interval, namely at the point $x=x_0$, then

$$\lim_{n=+\infty} \frac{\int_a^b f(x)\phi(x)^n dx}{\int_a^b \phi(x)^n dx} = f(x_0).$$

The theorem is closely related to a theorem of Stieltjes,* and the proof is based on methods similar to that used in Graeffe's method for determining the roots of an algebraic equation with numerical coefficients.

20. Ruled surfaces with isotropic generators have been considered by Monge, J. A. Serret, Lie, and others. Professor Eisenhart uses for the determination of such a surface the curve in which it is cut by a plane and the directions of the lines which are the projections on the plane of the generators of the surface. In this way a ruled surface of this type is determined by a set of lineal elements, in a plane, depending upon one parameter. The envelope of the direction of the lineal elements is a generalization of oblique evolutes, or evolutoids. Such a surface is determined also by a curve in space, in the sense that it is one of the two sheets of the envelope of a certain family of spheres whose centers are on the space curve. The equations of the two sheets assume a symmetrical form when the space curve is defined in two ways in terms of the two "normal parameters," as defined by the author in the September number of the *Annals*. The relations between the space curve and the set of lineal elements on the plane are very interesting, and lead to a significant transformation of sets of plane curves. Of particular interest is the case in which the spheres are of constant radius, so that the Gaussian curvature of the surface is a positive constant, and thus the ruled surface is a deform of a sphere in which one set of numerical generators remain right lines.

21. In his first paper Professor Young uses the essential ideas of von Staudt's algebra of points on a line to develop the general notion of an algebra associated with certain groups

* Correspondance d'Hermite et de Stieltjes, vol. 2 (1905), p. 185.

of transformations. He considers first a group $\mathfrak{G}$ of transformations which operates in a simply transitive way on the elements $a, b, c, \cdots$ of a certain class Σ. By choosing any particular element of Σ, say i, to be the so-called identical element, there exists in $\mathfrak{G}$ a unique transformation G_a, such that $G_a(i) = a$. An operation $\circ$ acting on pairs of elements of $\mathfrak{G}$ is then defined by the relation $a \circ b = G_a(b)$. The operation $\circ$ is then associative; it is commutative if $\mathfrak{G}$ is commutative; the element i satisfies the relations $a \circ i = i \circ a = a$ for every a of Σ; and corresponding to every a in Σ there exist in Σ two unique elements a', a'' such that $a \circ a' = i$ and $a'' \circ a = i$. The group $\mathfrak{G}$ is then represented by the equations $x' = a \circ x$, the individual transformations of $\mathfrak{G}$ being obtained by allowing a to vary over Σ. The equations $x' = x \circ a$ define a group $\overline{\mathfrak{G}}$, simply isomorphic with $\mathfrak{G}$. $\overline{\mathfrak{G}}$, which is called the adjoint group of $\mathfrak{G}$, is identical with $\mathfrak{G}$ only when $\mathfrak{G}$ is commutative. The author considers next two groups, say $\mathfrak{A}$ and $\mathfrak{M}$, acting in a simply transitive way on two classes Σ_1 and Σ_2 respectively. The operations defined as above by $\mathfrak{A}$ and $\mathfrak{M}$ are for convenience called "addition" and "multiplication" and denoted by $+$ and $\cdot$ respectively; the identical element with respect to $+$ is denoted by 0 and the identical element with respect to $\cdot$ by 1. If Σ_1, Σ_2 have a non-empty class Σ in common, there will be relations between addition and multiplication determined by the relations between the groups $\mathfrak{A}$ and $\mathfrak{M}$. If, in particular, $\mathfrak{M}$ transforms $\mathfrak{A}$ into itself, there is obtained a generalized law of distributivity, expressed by the following relation: $a \cdot (b+c) = a \cdot b + (a \cdot 0)' + a \cdot c$, where the notation m' is used for the element such that $m' + m = 0$. The ordinary form of distributivity is obtained if $a \cdot 0 = 0$ for all values of a; i. e., if every transformation of $\mathfrak{M}$ transforms 0 into itself (0 is then not an element of Σ_2; it is exceptional for $\mathfrak{M}$). The distributive relation just given implies the commutativity neither of addition nor of multiplication. The condition $\mathfrak{M}\mathfrak{A}\mathfrak{M}^{-1} = \mathfrak{A}$, leads to general left-handed distributivity. The condition for general right-handed distributivity is $\overline{\mathfrak{M}}\mathfrak{A}\overline{\mathfrak{M}}^{-1} = \mathfrak{A}$, where $\overline{\mathfrak{M}}$ is the adjoint group of $\mathfrak{M}$. Von Staudt's algebra of points on a line is obtained by letting $\mathfrak{A}$ be the group of all parabolic projectivities having a common double point M, and letting $\mathfrak{M}$ be the group of all projectivities having two common double points M and O, where O is the identical element with respect to $\mathfrak{A}$. The author discusses several other examples which lead to new analytic representations of certain groups of projective and birational transformations.

22. In his second paper, Professor Young employs the methods developed in his first paper to define a point algebra in a space S_n of n dimensions as follows: Multiplication is defined by the group $\mathfrak{M}$ of collineations in S_n leaving the vertices $O, U_1, U_2, \cdots, U_n$ of a complete $(n + 1)$-point invariant, $a \cdot b$ being by definition the point into which b is transformed by the transformation M_a of $\mathfrak{M}$ which transforms the unit point 1 into a. Addition is defined by the group $\mathfrak{A}$ of elations in S_n leaving fixed every point of the S_{n-1} determined by $U_1, \cdots, U_n$, $a + b$ being the point into which b is transformed by the transformation A_a of $\mathfrak{A}$ which transforms the zero point $O = 0$ into a. Multiplication and addition are then associative and commutative, and multiplication is distributive with respect to addition. In fact, the points of S_n form a field with respect to these two operations, the exceptional elements being the points of the n $(n - 1)$-spaces of the fundamental $(n + 1)$-point $O, U_1, \cdots, U_n$ for multiplication and the points of the S_{n-1} determined by $U_1, \cdots, U_n$ for addition. The author then considers the group $\mathfrak{A}_n$, of $3n$ parameters, consisting of all transformations $x' = \dfrac{ax + b}{cx + d}$, $ad - bc$ not a divisor of zero. $\mathfrak{A}_1$ is then the general projective group on a line; $\mathfrak{A}_2$ is the group of all quadratic transformations in a plane having two fundamental points U_1 and U_2 in common, and is therefore equivalent to the group of direct circular transformations in a plane first discussed by Möbius; $\mathfrak{G}_3$ is a group of Cremona transformations in S_3 of order 3 having three fundamental points U_1, U_2, U_3 in common; in general $\mathfrak{G}_n$ is a group of Cremona transformations of order n in S_n. The representation of the group given above makes it possible to apply methods of the projective geometry on a line to the geometry defined by $\mathfrak{G}_n$. The elementary curves of the geometry, the so-called C-curves, are the curves equivalent under $\mathfrak{G}_n$ to the straight line through 0 and 1. There is one and only one such curve through any 3 points a, b, c, provided no two of the latter are coplanar with two of the points U_i. The class of C-curves contains all the normal

curves of degree n in S_n, passing through the n points U_i $(i = 1, \cdots, n)$ and certain normal curves of lower orders. For example, for $n = 3$ the C-curves consist of all twisted cubics through U_1, U_2, U_3, of all conics through one of the points U_i and meeting the opposite side of the triangle $U_1U_2U_3$, and of all straight lines in S_3 not on the plane $U_1U_2U_3$. Any transformations of $\mathfrak{G}_n$ of general type has 2^n double points, which are associated in a definite way into 2^{n-1} pairs. Among the theorems derived may be mentioned the following: Any involution (of general type) in $\mathfrak{G}_n$ leaves invariant every C-curve through every pair of associated double points, and only these. If a transformation of $\mathfrak{G}_n$ leaves one C-curve invariant, it leaves one, and if the transformation is non-involutoric only one, such curve through every point not a double point invariant; etc. The author also discusses the extension of the group $\mathfrak{G}_n$ by the adjunction of transformations of another kind, which reduces in the case $n = 2$ to the introduction of the indirect circular transformations.

23. A series is absolutely convergent, if the series formed from the absolute values of its terms converges; unconditionally convergent, if it converges for every order in which its terms are arranged. The two definitions are equivalent for convergent series. In this note, Dr. Silverman proposes to inquire whether similar notions are possible in the theory of summable series.

The notion of absolute convergence is obviously not generalizable to the case of summable series,* if we restrict† ourselves to those definitions for which it is true that whenever $\lim(S_n) = s$ or ∞, then $G(S_n) = s$ or ∞ respectively, where lim stands for the ordinary limit, and G for the generalized limit. That the notion of unconditional convergence, too, cannot be generalized to the case of summable series, if we restrict ourselves to the same definitions as above, results from the following theorem: If corresponding to every arrangement of the terms of a series there is some definition of summability which gives it a value (possibly different in each case), then the series is absolutely convergent.

* Borel has used the term absolute summability, but in a different sense from that considered here. See Leçons sur les Séries divergentes, p. 99.

† This restriction excludes such definitions as that of absolute summability given by Borel; but this definition of Borel's is not a true generalization of convergence, as has been pointed out by Hardy, *Quarterly Journal*, vol. 35 (1903), p. 25.

24. In this note Dr. Silverman considers some simple sufficient conditions for Cesàro summability, similar to some of the well-known tests for convergence.

25. Professor Fite calls attention to a collineation group in n variables that seems not to have been noticed before. It is of order $n^3 b^{a-1} d^{c-1}/f$, where $n = ab = cd$, a and b relatively prime, c relatively prime with both a and d, $a < n$, $c < n$, and f the greatest common divisor of $c - a$ and d. It contains an invariant abelian subgroup of index n.

26. Professor Eisenhart gives the following analytical definition of a congruence of minimal lines in euclidean four-space: Given a congruence C of non-minimal lines in ordinary space, each line being defined by the coordinates (x_0, y_0, z_0) of the point P_0 in which the line meets a surface of reference S referred to curvilinear coordinates u, v and by its direction cosines X, Y, Z, where x_0, y_0, z_0 and X, Y, Z are one-valued continuous functions of u and v, admitting continuous first derivatives; if x, y, z are the coordinates of a point P on this line and t the distance P_0P, then the equations

$$x = x_0 + tX, \quad y = y_0 + tY, \quad z = z_0 + tZ, \quad w = it$$

define a congruence Γ of minimal lines in four-space of the most general type. The condition that a one-parameter system of these lines, determined by a relation $f(u, v) = 0$, be tangent to a curve, i. e., developable two-spread, is that the corresponding ruled surface of the congruence C, which is defined by the first three of the above equations, be a developable surface, and that it meets S in an orthogonal trajectory of the generators of this developable. When C is a normal congruence, and only in this case, through each line of Γ there pass two developables of the congruence. (This is the particular type of minimal congruences recently considered by Eiesland in the *Transactions*.) However, any surface S and a family of curves upon it leads to a congruence Γ such that through each line of Γ there passes one developable. But for the general congruence Γ none of the ruled two-spreads is developable. The paper deals with other properties of congruence Γ.

F. N. COLE,
Secretary.