ON A FUNCTIONAL EQUATION.

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1. ABEL,* in a discussion rigorized by Stäckel,† has shown (in effect) that if

(1) $$f(x, u) = f(y, v) = f(z, w),$$

where

(2) $$f(y, z) = u, \quad f(x, z) = v, \quad f(x, y) = w,$$

then there exists a function χ, dependent upon the function f, such that

$$\chi(f(x, y)) = \chi(x) + \chi(y).$$

In this paper it is desired to establish the following theorem: *If $f(v, w) = u$,‡ then there exists a function χ such that*

$$\chi(f(x, y)) = \chi(x) - \chi(y).$$

In a sense, this theorem is a correlative of the theorem of Abel.

2. The method of proof of the preceding theorem resembles the procedure of Abel and Stäckel, l.c. Differentiating the relation

(3) $$f(v, w) = u$$

with respect to y, z, x, we have

$$0 + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y} - \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + 0 - \frac{\partial u}{\partial z} = 0,$$

$$\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x} - 0 = 0.$$

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* *Crelle's Journal*, vol. 1 (1826), p. 11; cf. M. Cantor, *Zeitschrift für Mathematik und Physik*, vol. 41 (1896), p. 161.

† *Zeitschrift für Mathematik und Physik*, vol. 42 (1897), p. 323; cf. P. Hayashi, same journal, vol. 44 (1899), p. 346.

‡ This relation, with the preceding definitions of u, v, w, we call quasi-transitiveness. Abel, l.c., bases his discussion on the symmetry of $f(z, w)$ in x, y, z.

Therefore, on elimination of $\partial f/\partial v$, $\partial f/\partial w$,

$$\begin{vmatrix} 0 & \partial w/\partial y & \partial u/\partial y \\ \partial v/\partial z & 0 & \partial u/\partial z \\ \partial v/\partial x & \partial w/\partial x & 0 \end{vmatrix} = 0$$

or

$$\frac{\partial v}{\partial x}\cdot\frac{\partial w}{\partial y}\cdot\frac{\partial u}{\partial z} = -\frac{\partial v}{\partial z}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial u}{\partial y}. \tag{4}$$

The relation (4) is, of course, much more general than relation (3) from which it is derived, since (4) can be obtained, for instance, by differentiating any one of the relations*

$$\phi(v, w) = u, \quad \phi(w, u) = v, \quad \phi(v, u) = w.$$

Relation (4) differs, moreover, only in sign from a relation derived from (1) by Abel and Stäckel, and hence proceeding in the manner of the latter, we obtain from (4), if z = constant,

$$\frac{\partial w}{\partial y}\cdot\psi'(x) = -\frac{\partial w}{\partial x}\cdot\psi'(y) \tag{4'}$$

and as a solution† of (4′),

$$f(x, y) = \Omega(\psi(x) - \psi(y)),$$

where Ω remains to be further conditioned. Substituting in (3), we get

$$\Omega[\psi\Omega(\xi - \zeta) - \psi\Omega(\xi - \eta)] = \Omega(\eta - \zeta), \tag{5}$$

where

$$\psi(x) = \xi, \quad \psi(y) = \eta, \quad \psi(z) = \zeta.$$

On differentiating (5) with respect to y and z, and comparing the resulting relations, we find

$$\frac{\partial}{\partial\eta}\psi\Omega(\xi - \eta) = \frac{\partial}{\partial\zeta}\psi\Omega(\xi - \zeta). \tag{6}$$

Now we differentiate (6) with respect to y; then since

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* It may perhaps be remarked in passing that $f(v, w) = u$ follows from $\phi(w, u) = v$ and $f(\phi(x, y), x) = y$.

† Cf. Abel, *Crelle's Journal*, vol. 2 (1827), p. 389. See also Mansion-Maser, Partielle Differentialgleichungen, Berlin (1892), pp. 34–37.

$$\frac{\partial}{\partial \eta}\psi\Omega(\xi - \eta) = -\frac{\partial}{\partial(\xi - \eta)}\psi\Omega(\xi - \eta)$$

we have

$$\frac{\partial^2}{\partial(\xi - \eta)^2}\psi\Omega(\xi - \eta)\cdot\frac{\partial \eta}{\partial y} = 0.$$

If $\partial\eta/\partial y = 0$, then $\psi(y) =$ constant, and hence $f(x, y) =$ constant, which we exclude; therefore

$$\frac{\partial^2}{\partial(\xi - \eta)^2}\psi\Omega(\xi - \eta) = 0$$

or

(7) $$\psi\Omega(p) = cp + c_1, \quad p = \xi - \eta,$$

where c and c_1 are constants. Now let

(8) $$\chi(x) = \psi(x) - c_1;$$

then

$$\chi(\Omega p) = cp,$$

and consequently,

(9) $$\chi(f(x, y)) = c(\chi(x) - \chi(y)).$$

The constant c can be determined by substituting in (9) in accordance with (3); we find $c^2 - c = 0$, that is, since $f(xy) \neq$ constant, $c = 1$. Hence the theorem under § 1 is verified. Conversely, if

(10) $$f(x, y) = \chi^{-1}(\chi(x) - \chi(y)),$$

then $f(x, y)$ has the property (3). If $f(x, y)$ is assigned, then by following Abel, l. c., volume 1, page 14, the function χ can be determined by the formula

(11) $$\chi(x) = -\chi'(y)\cdot\int\frac{\partial w/\partial x}{\partial w/\partial y}dx,$$

where y is to be regarded as a constant. From this determination we deduce

1°. If $f(x, y) = \phi(x/y)$, then

$$\log c\cdot\phi(x/y) = \log cx - \log cy$$

and therefore, $\phi(x/y) = c'\cdot x/y$.

2°. If $f(x, y) = \phi(x - y)$, then $\phi(x - y) = x - y + c''$, where c' and c'' are constants.

In the case of the equation

$$(12) \qquad f(v, u) = w$$

a discussion similar to the above is valid. It may be observed that relation (12) follows from (3) and (10), as does also a part of relation (1), viz.,

$$f(y, v) = f(z, w).$$

3. We conclude with a few remarks concerning the nature of the generalization of relation (3), which is perhaps less obvious than the corresponding generalization of relation (1) of Abel. Presumably the relation (3) occurs in a sequence of functional relations

$$f(f(t_1, x_2), f(t_1, x_1)) = f(x_1, x_2),$$

$$f(f(t_1, t_2, x_3), f(t_1, t_2, x_2), f(t_1, t_2, x_1)) = f(x_1, x_2, x_3),$$

$$f(f(t_1, t_2, t_3, x_4), f(t_1, t_2, t_3, x_3), f(t_1, t_2, t_3, x_2), f(t_1, t_2, t_3, x_1)) = f(x_1, x_2, x_3, x_4),$$

etc. This sequence leads to the sequence of linear partial differential equations of the first order

$$\frac{\partial u_2^{(0)}/\partial x_2}{\psi_2'(x_2)} + \frac{\partial u_2^{(0)}/\partial x_1}{\psi_2'(x_1)} = 0,$$

$$\frac{\partial u_3^{(0)}/\partial x_3}{\psi_3'(x_3)} + \frac{\partial u_3^{(0)}/\partial x_2}{\psi_3'(x_2)} + \frac{\partial u_3^{(0)}/\partial x_1}{\psi_3'(x_1)} = 0,$$

$$\frac{\partial u_4^{(0)}/\partial x_4}{\psi_4'(x_4)} + \frac{\partial u_4^{(0)}/\partial x_3}{\psi_4'(x_3)} + \frac{\partial u_4^{(0)}/\partial x_2}{\psi_4'(x_2)} + \frac{\partial u_4^{(0)}/\partial x_1}{\psi_4'(x_1)} = 0,$$

etc., where

$$f(x_1, x_2) = u_2^{(0)}, \quad f(x_1, x_2, x_3) = u_3^{(0)}, \quad f(x_1, x_2, x_3, x_4) = u_4^{(0)},$$

etc. The preceding equations have respectively the solutions,*

$$u_2^{(0)} = F_2(\psi_2(x_1) - \psi_2(x_2)),$$

$$u_3^{(0)} = F_3(\psi_3(x_1) - \psi_3(x_2), \quad \psi_3(x_1) - \psi_3(x_3)),$$

$$u_4^{(0)} = F_4(\psi_4(x_1) - \psi_4(x_2), \quad \psi_4(x_1) - \psi_4(x_3), \quad \psi_4(x_1) - \psi_4(x_4)),$$

etc., where F_2, F_3, F_4, etc., are arbitrary functions to be conditioned by the original functional relations.

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* Mansion-Maser, l.c.