IMPLICIT FUNCTIONS DEFINED BY EQUATIONS WITH VANISHING JACOBIAN.

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LET

$$x_i = f_i(y_1, \cdots, y_n) \qquad (i = 1, 2, \cdots, n)$$

be n functions, each analytic in the neighborhood of the point $(y) = (0)$. The case in which the Jacobian

$$J = \sum \pm \frac{\partial f_1}{\partial y_1} \frac{\partial f_2}{\partial y_2} \cdots \frac{\partial f_n}{\partial y_n}$$

does not vanish in the point $(y) = (0)$, and the case in which J vanishes identically in the neighborhood of this point, are familiar. I wish to state here some theorems which I have obtained for the intermediate case that J vanishes in the point $(y) = (0)$, but does not vanish identically in the n variables y. For a number of the theorems n is restricted to the value 2.

If $x_1, \cdots, x_n$ denote n independent complex variables, any region which constitutes at least the totality of all points $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, \; | x_2 - a_2 | < \epsilon, \; \cdots, \; | x_n - a_n | < \epsilon$$

for some $\epsilon > 0$, is called a *complete* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

Any region which satisfies the conditions that

(1) for every $\epsilon > 0$, it contains points $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, \; | x_2 - a_2 | < \epsilon, \; \cdots, \; | x_n - a_n | < \epsilon;$$

(2) for no $\epsilon > 0$, *every* point $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, \; | x_2 - a_2 | < \epsilon, \; \cdots, \; | x_n - a_n | < \epsilon$$

is a point of the region

is called a *partial* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

For purposes of simplicity, the neighborhood of the point $(0, 0, \cdots, 0)$ is considered throughout this paper. In what follows, the unqualified word *neighborhood* is used to mean *complete neighborhood.*

Consider the transformation

$$T: \qquad x = f(u, v), \quad y = \varphi(u, v),$$

(*a*) $f(u, v)$ and $\varphi(u, v)$ denoting functions of the complex variables u and v, single-valued and analytic throughout a neighborhood R of $u = 0, v = 0$, and satisfying the conditions

$$(b) \qquad f(0, 0) = 0, \quad \varphi(0, 0) = 0,$$

$$(c) \qquad J(u, v) \equiv \begin{vmatrix} f_u & f_v \\ \varphi_u & \varphi_v \end{vmatrix} \not\equiv 0, \quad J(0, 0) = 0.$$

To any point (u, v) of R, there corresponds, under T, one and but one point (x, y) of the (x, y) space. The totality of points (x, y) which thus correspond to points (u, v) of R, constitute a finite region $\bar{R}$ of the (x, y) space. In general, more than one point (u, v) will yield the same pair of values for x and y. We shall count a point (x, y) but once, thus regarding it as completely characterized by its coordinates, and seek an inverse transformation

$$u = \bar{f}(x, y), \quad v = \bar{\varphi}(x, y)$$

which will put in evidence all those points (u, v) of R which correspond under T to any point with coordinates (x, y) of $\bar{R}$. Thus the functions $\bar{f}(x, y)$, $\bar{\varphi}(x, y)$ are to be defined only in the points of $\bar{R}$, and there uniquely or as multiple-valued functions.

THEOREM I. *The transformation T can never have an inverse*

$$u = \bar{f}(x, y), \quad v = \bar{\varphi}(x, y)$$

such that $\bar{f}(x, y)$ and $\bar{\varphi}(x, y)$ are both analytic throughout a complete neighborhood of $x = 0$, $y = 0$.

The theorem holds for any value of n.

THEOREM II. *If in the transformation T, $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$, then there exists an inverse, defined throughout a complete neighborhood of $x = 0$, $y = 0$, everywhere continuous, finitely multiple-valued, analytic except along a complex one-dimensional locus, and having the value $u = 0$, $v = 0$, when $x = 0$, $y = 0$.*

If instead of the transformation T, we consider the transformation T', whose definition differs from that of T in that it omits the condition (c), we deduce

THEOREM III. *If to each point (x, y) of the region $\bar{R}$ there corresponds from T' a single point (u, v) of a region R, then u and v are analytic in x and y in the point $x = 0, y = 0$.*

It follows from Theorem I and from Theorem III that there can not exist a transformation T for which the inverse is single-valued throughout any neighborhood $\bar{R}$ of the point $x = 0, y = 0$.

The proof for Theorem II and for Theorem III, depends on the Weierstrassian implicit function theorem.* Theorem IV is a generalization of this.

THEOREM IV. *The system of equations*

$$f_1(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,$$
$$f_2(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$f_p(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,$$

where

(1) *f_i is an analytic function of its arguments throughout a neighborhood R of $(x) = 0, (y) = 0$;*

(2) $f_i(0, \cdots, 0; 0, \cdots, 0) = 0 \quad (i = 1, 2, \cdots, p)$;

$$(3)\quad J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \textit{ when } (x) = 0, (y) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \textit{ when } (x) = 0, (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \textit{ when } (x) = 0, (y) = 0,$$

defines $y_1, y_2, \cdots, y_p$ as functions of $x_1, x_2, \cdots, x_n$, continuous throughout the neighborhood of $(x) = 0$. With a suitable counting of multiplicities everywhere present, this solution is k-fold; it is analytic with the possible exception of complex $(n-1)$-dimensional loci where determinations of the root, in general distinct, become coincident. When $(x) = 0, (y) = 0$.

* Weierstrass, Abhandlungen aus der Funktionenlehre, p. 105; Bliss, BULLETIN, April, 1910, p. 356; Macmillan, ibid., December, 1910, p. 116.

THEOREM V. *The transformation*

$$x_1 = f_1(y_1, \cdots, y_n), \quad x_2 = f_2(y_1, \cdots, y_n), \cdots, x_n = f_n(y_1, \cdots, y_n),$$

where

(1) f_i *is a single-valued and analytic function of* $y_1, \cdots, y_n$ *in the point* $(0, \cdots, 0)$ *and vanishes there* $(i = 1, 2, \cdots, n)$;

$$(2) \qquad J_1(0, \cdots, 0) = \left.\frac{D(f_1, \cdots, f_n)}{D(y_1, \cdots, y_n)}\right|_{(y)=0} = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$J_{k-1}(0, \cdots, 0) = \left.\frac{D(J_{k-2}, f_2, \cdots, f_n)}{D(y_1, y_2, \cdots, y_n)}\right|_{(y)=0} = 0,$$

$$J_k(0, \cdots, 0) = \left.\frac{D(J_{k-1}, f_2, \cdots, f_n)}{D(y_1, y_2, \cdots, y_n)}\right|_{(y)=0} \neq 0,$$

has a k-valued continuous inverse, defined throughout a complete neighborhood of $(x) = 0$. *This inverse is analytic, with k distinct determinations, except along a complex* $(n-1)$*-dimensional locus, where it is continuous and less than k-valued. When* $(x) = 0$, $(y) = 0$.

The remaining theorems have to do with a detailed study of the transformation T. Theorems VI, VII, and VIII yield a complete discussion for the case in which at least one element of the determinant J, say f_u, is not zero at the origin.

THEOREM VI. *If the transformation T satisfies the further conditions*

(1) $f_u(0, 0) \neq 0$,

(2) $f(u, v)$ *not a factor of* $\varphi(u, v)$ *in the point* $u = 0$, $v = 0$, *then, for the sequence of functions*

$$J_1(u, v) \equiv \frac{D(f, \varphi)}{D(u, v)}, \cdots, J_{i+1}(u, v) \equiv \frac{D(f, J_i)}{D(u, v)}$$

there exists an integer $k \geqq 2$ *such that*

$$J_1(0, 0) = \cdots = J_{k-1}(0, 0) = 0, \quad J_k(0, 0) \neq 0,$$

and for the transformation there exists a k-valued continuous inverse, defined throughout a complete neighborhood of $x = 0$, $y = 0$. *This inverse is analytic, with k distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than k-valued. Finally* $u = 0$, $v = 0$ *when* $x = 0$, $y = 0$.

THEOREM VII. *If the transformation T is of the form*

$$x = f(u, v), \quad y = [f(u, v)]^n g(u, v),$$

where

$$f_u(0, 0) \neq 0, \quad g(0, 0) = 0,$$

and where $f(u, v)$ is not a factor of $g(u, v)$ in the point $u = 0$, $v = 0$, then to the point $x = 0, y = 0$ corresponds the locus $f(u, v) = 0$, and to any other point (x, y)

(1) *if $f_u g_v - f_v g_u \neq 0$ when $u = 0$, $v = 0$, there corresponds one and only one point (u, v), the relationship being defined by functions of x and y, single-valued and analytic in this point;*

(2) *if $f_u g_v - f_v g_u = 0$ when $u = 0$, $v = 0$, there correspond in general m points, $m \geqq 2$, the relationship being defined by m-valued continuous functions of x and y.*

THEOREM VIII. *A transformation T for which $f(u, v)$ is a factor of $\varphi(u, v)$, can be replaced by a transformation*

$$u_1 = f(u, v), \quad v_1 = h(u, v),$$

where $h(u, v)$ is a single-valued and analytic function of u and v in the point $(0, 0)$ and vanishes there, and where $f(u, v)$ is not a factor of $h(u, v)$ in this point, followed by a finite number m of transformations of the form

$$x = u, \quad y = uv,$$

combined with not more than m translations of the form

$$x = u, \quad y = v + c.$$

THEOREM IX. *A transformation T which has the form*

$$x = f(u) = a_n u^n + a_{n+1} u^{n+1} + \cdots \quad (a_n \neq 0),$$
$$y = \varphi(u, v),$$

can be replaced by two transformations, $T = b \cdot a$, where

$$\begin{array}{lll} a: & u_1 = u \sqrt[n]{a_n + a_{n+1} u + \cdots}, & v_1 = \varphi(u, v), \\ b: & x = u_1^n, & y = v_1. \end{array}$$

The inverse of (a) is defined by analytic functions, or can be determined from Theorems VI, VII *and* VIII.

THEOREM* X. *If $k = 2$ in Theorem* VI, *the transformation*

* Cf. L. S. Dederick, Harvard doctoral thesis (1909), "Certain singularities of transformations of two real variables," p. 124.

T can be replaced by three transformations which are one-to-one and analytic both ways, combined with one transformation of the form

$$x = u, \quad y = v^2.$$

THEOREM XI. *If the transformation T has the form*

$$x = f(u, v) \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + \cdots,$$

$$y = \varphi(u, v) \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + \cdots,$$

where the terms quadratic in u and v are not identically zero for either f or φ, and where these quadratic terms have no common factor, then there exists a four-valued continuous inverse, defined throughout the complete neighborhood of $x = 0$, $y = 0$. This inverse is analytic, with four distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than four-valued. Finally, $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

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DARWIN'S SCIENTIFIC PAPERS.

Scientific Papers. By SIR GEORGE HOWARD DARWIN, K.C.B., F.R.S., Plumian Professor in the University of Cambridge. Vol. III, Figures of Equilibrium of Rotating Liquid and Geophysical Investigations, xv + 527 pp.; Vol. IV, Periodic Orbits and Miscellaneous Papers, xviii + 592 pp. Cambridge University Press, 1910, 1911. Royal 8vo.

THE first two volumes of Sir George Darwin's researches have already been reviewed in the columns of the BULLETIN.* They contained papers on the practical and theoretical tidal problems which the oceans present and his earlier attacks on the past history of the earth-moon system. The third and fourth volumes contain his investigations on the relations of fluid masses in rotation about an axis under gravitational forces, on the periodic orbits which a particle can describe when attracted by two bodies of finite masses moving in circular orbits about one another, and a number of papers on other matters.

Of the forms which a single mass of fluid can take when revolving without relative motion about an axis under its own gravitational attraction only, two have long been known. Maclaurin's ellipsoid is one of revolution about the axis of rotation and its eccentricity will have a value which depends

* Vol. 16, pp. 73–78.