in all probability true for the manifolds ∞^{nf_i}, ∞^{mf_i}, $n > m$; and for ∞^{f_i}, ∞^{f_j}, $i > j$; but where is a proof to be found? The question is of interest both for analytic functions and for general continuous functions.

EDWARD KASNER.

Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft: *Ebene analytische Kurven und zu ihnen gehörige Abbildungen.* Von E. STUDY. Leipzig, Teubner, 1911. 126 pp. M. 4. 80.

As the title indicates, this monograph contains the first part of a series of lectures on a number of selected geometric topics and deals with the geometry in the complex domain, defined as a cartesian plane with ordinary complex numbers as coordinates. Points of such a domain are, for example, imaginary points of intersection of algebraic curves.

In the introduction we find a summary of von Staudt's famous treatment of imaginary elements by elliptic involutions. The disadvantage of this method is that in order to apply it to the solutions of even some very simple problems concerning positional relations a very clumsy apparatus has to be set in motion.

It is therefore desirable to devise a scheme by which problems involving imaginaries can be handled in a simpler and more effective manner. This is exactly what Study accomplishes in a very thorough manner in his valuable monograph.

He starts out by defining the first and second picture (erstes und zweites Bild) of an imaginary point (ξ, η) in a plane. Designating by $\bar{\xi}$ and η the conjugates of ξ and η, the first picture is obtained as the real pair of points of intersections of the left and right handed minimal lines through (ξ, η) and $(\bar{\xi}, \bar{\eta})$. As indicated in a footnote on page 10, this representation is (apparently) due to Laguerre. It is well to state at this point explicitly that as a pioneer in the field of the complex domain Laguerre accomplished as much as any other man and may be justly put on the same level with von Staudt. As early as 1853 Laguerre found the remarkable result that the radian measure of an angle may be defined as the product of $\frac{1}{2}\sqrt{-1} = \frac{1}{2}i$ and the cross-ratio formed by the two sides of the angle and the isotropic lines through its vertex. In two further articles "Sur l'emploi des imaginaires en géométrie," which in 1870 also appeared in the *Nouvelles Annales de Mathématiques,* he laid the foundation of the theory which Study so aptly calls geometry in the complex domain.

In this connection also, the author ought not forget to mention the valuable contribution of Darboux in Sur une Classe remarquable de Courbes et de Surfaces,* in which by means of the "points associés," † corresponding to Study's "first picture," he established the focal properties of conics and generalized cassinians and lemniscates.

As the "second picture" (zweites Bild) Study introduces a second couple of real points (Z, W) on the line joining (ξ, η) with $(\bar{\xi}, \bar{\eta})$, which have the same middle point as the first real couple (z, w) and whose distance differs from the purely imaginary distance $\sqrt{(\xi - \bar{\xi})^2 + (\eta - \bar{\eta})^2}$ by a primitive fourth root of unity. According to Study, this idea also is not new and may be traced back to a number of independent investigators. The credit of being the first to make a systematic application of these representations to geometry is according to Study due to Segre.‡ Study however follows an entirely independent path leading to a great number of original views and results.

Without entering into a detailed account of the rest of the subject matter, the nature and scope of the book will perhaps best appear from an explicit statement of a few propositions.

The points, z, w, Z, W form the vertices of a square. Evidently the segment $\overrightarrow{ZW}$ may be obtained from the segment $\overline{zw}$ by turning the latter about its middle point through a positive angle $\frac{1}{2}\pi$. This periodic process which, after repeating it four times, leads to identity is called "positive change of front" (positive Schwenkung) of the couple $\overrightarrow{zw}$. The process

* A. Hermann, Paris, 1st ed. 1873, 2d ed. 1896. Davis, in the University of Nebraska Studies, vol. 10, pp. 1–58 (1910) also gives a method establishing a (1, 1) correspondence between imaginary points and real elements. Recently Mathews in *Proc. London Math. Society,* vol. 10, part 3, pp. 173–190 (1911) has established a similar method.

† It can easily be shown that an imaginary point, its conjugate, and their associated real couple (erstes Bild, points associés) may be represented by the intersections of two orthogonal equilateral hyperbolas. In other words every imaginary point may be represented by two real orthogonal equilateral hyperbolas. The four intersections form an orthogonal quadrangle with the circular points as two of the diagonal points.

‡ "Un nuovo campo di ricerche geometriche," *Atti di Torino,* vol. 25, pp. 35–71, vol. 26, pp. 276–301, 430–457, 592–612 (1899).

which results from three repetitions he calls "negative change of front" (negative Schwenkung). The relations between these couples lead incidentally to some interesting kinematic propositions:

If two opposite vertices of a variable square move with constant velocities on two straight lines, then the same is true of the other vertices.

If two opposite vertices of a variable square move in the same sense on two circles with velocities proportional to the radii, then the same is true of the other vertices. The middle points of the four circles also form a square.

The principal object is, of course, the investigation of analytic curves by means of the real pictures. An analytic curve consists of all complex points and only of such points (with finite coordinates ξ_0, η_0) in whose neighborhood the dependence of ξ and η can be expressed at least by one of the developments

$$\eta - \eta_0 = \mathfrak{P}((\xi - \xi_0)^{\frac{1}{m}}) \qquad (m = 1, 2, \cdots),$$

$$\xi - \xi_0 = \mathfrak{P}((\eta - \eta_0)^{\frac{1}{n}}) \qquad (n = 1, 2, \cdots),$$

where the $\mathfrak{P}$'s represent convergent power series with positive integral or fractional exponents, without a constant term.

Every complex point of an analytic curve may be represented by either its first or second picture. If this is done for all points of the curve, the first and second pictures of the curve result, whose properties appear from the theorems:

The first picture of an analytic curve is generally any real improper (uneigentlich) conformal transformation $z \rightarrow w$. The second picture is generally a special real proper transformation preserving areas (flächentreue Transformation).

As an analytic thread (Faden) Study defines with Segre the set of ∞^1 complex points whose picture in space of four dimensions is a real branch of an analytic curve. An analytic membrane is a set of ∞^2 complex points whose four-dimensional picture is a real sheet of an analytic surface.

In the succeeding articles the differential elements of arcs in the first and second picture, the points of special behavior of an analytic curve and its pictures are discussed. The ellipse and catenary serve as examples.

Finally a chapter is devoted to the analytic continuation of the potential function. It contains some valuable suggestions and pertinent remarks on the legitimate use of higher complex number systems. It is shown that the analytic continuation through the bicomplex domain, for instance, cannot yield anything new for the ordinary complex domain which cannot also be obtained by analytic continuation in the latter domain.

On the other hand it is pointed out that certain systems of complex quantities, which do not obey the commutative law of multiplication have been particularly useful for the applications in the theory of groups and in geometry. "Accordingly we must beware of any kind of dogmatism. Science disregards artificial restrictions and acknowledges only its own laws, but no authorities."

In conclusion it is shown that the entire projective geometry in space may be interpreted in the euclidean plane by means of the real point couples or pictures. A number of suggestions are also made, how the methods used by the author may be extended to space.

The little book, on the whole very carefully prepared, thus proves very profitable reading and suggestive for further research.

ARNOLD EMCH.

Vector Analysis. By J. G. COFFIN. Second edition. Wiley and Sons, New York, 1911. 12mo. xxii+262 pages. $2.50.

THE first edition of this book was reviewed in this BULLETIN, volume 17 (1910–11), page 101. The present edition differs little from the first, the important additions occurring in the examples and the appendices. Some few errors have been corrected, and a few statements reworded. Thus, we find the definition of vector now given as follows: "A vector is a directed segment of a straight line on which are distinguished an initial and a terminal point." Exactly what this new definition means, is hard to see. The last clause is superfluous if the main clause means anything, for a segment necessarily has end-points, and if "directed" one end is necessarily initial and the other terminal. Why the end points are specially important is not made clear. Further, the term vector as used in the text does not mean a segment of a straight line, but any one of an infinity of parallel segments of the same currency. It would therefore seem better to