THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS. SECTION I: ARITHMETIC, ALGEBRA, ANALYSIS.

PROFESSORS E. B. Elliott, E. Landau, É. Borel, E. H. Moore, and H. von Koch acted as chairmen. Following is the list of papers, with abstracts, so far as they have been furnished by the writers.

HARDY, G. H. and LITTLEWOOD, J. E.: "Some problems of Diophantine approximation."

DRACH, J.: "Sur l'intégration logique des équations différentielles."

MOORE, E. H.: "On the fundamental functional operation of a general theory of integral equations."

In this paper Professor Moore states the principal general theorems involving the functional operation J of the system Σ_5 of terms and postulates defined in his paper in the April, 1912, number of the BULLETIN. On the basis of these theorems one may, along the lines of Fredholm, Plemelj, Hilbert, E. Schmidt, Goursat, Landsberg and I. Schur, develop a general theory of linear integral equations which contains as instances current theories of the equations I, II_n, III, IV defined in the previous paper.

BERNSTEIN, S.: "Sur les recherches récentes relatives à la meilleure approximation des fonctions continues par des polynomes de degré donné."

SILBERSTEIN, L.: "Some applications of quaternions." In the absence of Professor Silberstein this paper was read by title.

MACFARLANE, A.: "On vector analysis as generalized algebra."

The ordinary algebraic quantities may be taken as numbers or as vectors having a common direction. The corresponding elements of the generalized algebra are vectors having any direction in space. In a sum the vectors are successive either in a line or in a cycle; their resultant is not the full equivalent of the complex. The paper investigates the consequent changes in the rules and processes of algebra including the general form of the binomial theorem. For infinitesimal successive vectors the complex is a curve; the rules for differentiation and integration are investigated and the generalized form of Leibniz' theorem is obtained, as well as the generalized formulas for the differential and ∇ of the nth order. This may be said to be the geometric part of generalized algebra. The trigonometric or transcendental part is analogous. The directed logarithm of the angle is a Hamiltonian vector, and the algebra of these logarithms is very largely analogous to that for the line vectors.

JOURDAIN, P. E. B.: "The values that certain analytic functions can take."

In the absence of Professor Jourdain, his paper was presented by Professor Hardy.

KÜRSCHÁK, J.: "Limesbildung und allgemeine Körpertheorie."

Study of K. Hensel's p-adic numbers led the author to introduce a new concept, a generalization of absolute value. To every element a of a field K let there be assigned a real number $\| a \|$ satisfying the conditions: (1) $\| 0 \| = 0$; if $a \neq 0$, then $\| a \| > 0$; (2) $\| 1 + a \| \leqq 1 + \| a \|$; (3) $\| ab \| = \| a \| \cdot \| b \|$; (4) K contains at least one element a for which $\| a \|$ is different from both zero and unity.

The number $\| a \|$ is called the "valuation" (Bewertung) of a. We always find $\| 1 \| = 1 = \| -1 \|$ and $\| a - b \| \geqq \| a - c \| + \| c - b \|$, whence $\| a - b \|$ is a special case of the Mengenlehre concept "Ecart." In order to carry over the concepts of limit and fundamental series we need only substitute Bewertung for absolute value in the usual definitions. If in an "evaluated" realm K every fundamental series has a limit, then K is called perfect. The object of the investigation is to show that by adjunction of new elements every evaluated realm can be enlarged into a perfect one, which, moreover, is algebraically closed. This object is attained by generalizing various investigations of G. Cantor, E. Steinitz, J. Hadamard, and K. Weierstrass.

BATEMAN, H.: "Some equations of mixed differences occurring in the theory of probability and the related expansions in series of Bessel's functions."

A set of objects whose number is constantly increasing are distributed among a large number of boxes, there being no restriction as to the number in each box. The objects are marked with the numbers ± 1 and it is an even chance that a particular object has the mark $+1$. When the average number of objects in a box is a the chance that a box, chosen at random, contains numbers adding up to n is supposed to be a known function $f(n)$. It is required to find the corresponding chance $F_n(x)$ for the case when the average number of objects has increased to x. It is found that when the number of boxes is treated as very large the function F_n satisfies the equation of mixed differences

$$\frac{dF_n}{dx} = \tfrac{1}{2}[F_{n-1} + F_{n+1} - 2F_n].$$

By aid of the particular solution $e^{-(x-a)} I_{n-m}(x-a)$, the general solution is obtained in the form

$$F_n(x) = e^{-(x-a)} \sum_{n=-\infty}^{\infty} I_{n-m}(x-a) f(m).$$

Interesting expansions in series of Bessel's functions $I_m(x)$ are obtained from this result by using particular solutions $F_n(x)$. Problems leading to the difference equations

$$\frac{dF_n}{dx} = F_{n-1} - F_n, \quad \frac{d^2F_n}{dt^2} = k^2(F_{n+1} + F_{n-1} - 2F_n)$$

are treated similarly. The last equation occurs in the writings of John Bernoulli, d'Alembert, and Euler.

PETROVITCH, M.: "Fonctions implicites oscillantes."

An extension of Sturm's theorem on the zeros of the integrals of a linear equation of the second order enables the author to state simple, practical rules capable of putting in evidence the oscillating character and frequency of oscillation of implicit functions, in particular of the integrals which, together with their derivatives, are real, finite, and continuous in a given interval of the independent variable for an infinity of ordinary simultaneous differential equations of whatever order.

Among these equations are some which we meet in problems of dynamics.

HADAMARD, J.: " Sur la série de Stirling."

The problem is to replace a divergent series by a convergent one possessing the same asymptotic properties. Two methods have been employed: (1) transformation into a definite integral (Borel's method); (2) the method of the present paper—adding to each term a correction *which must be inferior in magnitude (for $x = \infty$) to every term of the given series,* thus securing convergence in a way analogous to Mittag-Leffler's in his theorem on the representation of meromorphic functions by series of rational fractions.

The asymptotic expression of $\log\Gamma(x)$ is closely allied with Raabe's integral $\int_x^{x+1} \log\,\Gamma(x)$ and by using the results of a memoir of Lindelöf,* the difference between this integral and $\log\,\Gamma(x)$ was found to be equal to

$$\int_0^\infty \frac{\log\,(x+it) - \log\,(x-it)}{2i(e^{2\pi t}-1)}\,dt.$$

Expanding the numerator in powers of t, we get precisely Stirling's series. To replace that divergent development by a convergent one, we need only separate the integral into two parts

$$\int_0^\infty = \int_0^x + \int_x^\infty.$$

Then

$$\int_0^x \frac{\log\,(x+it) - \log\,(x-it)}{2i(e^{2\pi t}-1)}\,dt = \Sigma\frac{(-1)^{n-1}}{2n-1}\left(\frac{B_n}{4n} - \int_x \frac{t^{2n-1}}{e^{2\pi t}-1}\,dt\right).$$

Expanding $1/(e^{2\pi t}-1)$ in powers of $e^{-2\pi t}$ gives the desired result

$$\log\,\Gamma(x+1) = \log\sqrt{2\pi} - x + (x+\tfrac{1}{2})\log x + \Sigma(-1)^{n-1}\left(\frac{B_n}{(4n-1)4n} - \varphi_n(x)\right) + R,$$

* *Acta Soc. Fennicæ*, vol. 31 (1912).

where the φ_n are polynomials of degree n in both $1/x$ and $e^{-2\pi x}$, therefore *of an order of magnitude* (*for* $x = \infty$) *inferior to that of any power of* $1/x$; the series is convergent; and

$$R = \int_x^\infty \frac{\log(x+it) - \log(x-it) + e^{-\pi t}[\log(x+ite^{-\pi t}) - \log(x-ite^{-\pi t})]}{2i(e^{2\pi t}-1)}\,dt$$

decreases (as x increases) at least as $e^{-2\pi x}$, therefore more rapidly than any power of $1/x$.

Schlesinger, L.: "Ueber eine Aufgabe von Hermite aus der Theorie der Modulfunktionen."

Fields, J. C.: "Direct derivation of the complementary theorem from elementary properties of rational functions."

Frizell, A. B.: "Axioms of ordinal magnitudes."
In this paper the author submits a set of eighty axioms intended to cover all assumptions needed for postulating well ordered types of order. A corollary is that it is not possible to postulate all well ordered types.

Padoa, A.: "Une question de maximum ou de minimum."
Let

$$z = y_1 y_2 \cdots y_n,$$

where

$$y_r = a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{r\,n-1}x_{n-1} + a_{rn};$$

and put

$$n\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

We may always assume $\Delta \geqq 0$.

Denoting by A_{rs} (inclusive of sign, $+$ or $-$ according as $r+s$ is even or odd) the minor of a_{rs}, let us consider only the case in which

$$A = A_{1n}A_{2n}\cdots A_{nn} \neq 0$$

and the variables $x_1, x_2, \cdots, x_{n-1}$ take those values only for which

$$A_{rn}y_r \geqq 0.$$

By purely elementary considerations it is shown that z always reaches either a maximum or a minimum according as A is positive or negative, that in both cases this value of z is Δ^n / A and corresponds to the values

$$x'_s = \frac{1}{n}\left(\frac{A_{1s}}{A_{1n}} + \frac{A_{2s}}{A_{2n}} + \cdots + \frac{A_{ns}}{A_{nn}}\right)$$

of $x_1, x_2, \cdots, x_{n-1}$.

The theorem is applied to determine the interior point of a tetraedron the product of whose distances from the faces is a maximum.

STERNECK, R. VON: "Neue empirische Daten über die zahlentheoretische Funktion $\sigma(n)$."

F. Mertens established the relation $|\sigma(n)| \leqq \sqrt{n}$ for all values of n up to 10,000 and showed that a general proof of it would justify Riemann's conjecture that the zeros of $\zeta(s)$ all have the real part $\frac{1}{2}$.

The author showed (1897–1901) by tabulating $\sigma(n)$ up to $n = 500{,}000$ that, except for a few values in the neighborhood of $n = 200$, the quotient $\sigma(n) / \sqrt{n}$ oscillates in this interval between ± 0.46.

More recently, with the support of the Imperial Academy at Vienna, he has had 16 new values of σ computed, ranging from $n = 500{,}000$ to $n = 5{,}000{,}000$. This computation gives for $|\sigma(n)| / \sqrt{n}$ the values: 0.297, 0.270, 0.022, 0.237, 0.214, 0.140, 0.209, 0.133, 0.302, 0.175, 0.230, 0.063, 0.073, 0.097, 0.083, 0.315, thus increasing the probability that Riemann's guess will prove correct and furnishing reason to expect that the limits of oscillation for $\sigma(n) / \sqrt{n}$ will turn out to be the same up to $n = 5{,}000{,}000$ as in the smaller interval previously examined.

ELLIOTT, E. B.: "Some uses in the theory of forms of the fundamental partial fractions identity."

If $F(\varphi) \equiv (\varphi - a_1)(\varphi - a_2) \cdots (\varphi - a_n)$, where φ is a symbol of direct differential operation, $\varphi(x, y, z, \cdots, \partial / \partial x, \partial / \partial y, \partial / \partial z, \cdots)$, and $a_1, a_2, \cdots, a_n$ constants, and if u is a solution of the differential equation

(1) $$F(\varphi)u = f(x, y, z, \cdots),$$

the identity

$$u = \sum_{s=1}^{s=n} \left\{ \frac{1}{F'(a_s)} \cdot \frac{F(\varphi)}{\varphi - a_s} u \right\}$$

expresses u as a sum $u_1 + u_2 + \cdots + u_n$, where, for $s = 1, 2, \cdots, n$, u_s satisfies

$$(\varphi - a_s)u_s = \frac{1}{F'(a_s)} f(x, y, z, \cdots)$$

in virtue of (1). The parts u_s are obtained by *direct operation.* This fact has numerous applications in the theory of forms. Three of them are exhibited in this paper. In all the right-hand side of (1) is zero.

I. The general rational integral homogeneous isobaric function of degree i and weight w in the coefficients of $(a_0, a_1, \cdots, a_p)(x, y)^p$ is separated by direct operation into $w + 1$ parts.

II. The most general rational integral function of $a_1, a_2, \cdots, a_p$ of degree i is separated into $ip + 1$ parts, which are orthogonal invariants with the various possible multiplying factors $e^{\iota(ip-2m)}$, $\iota = \sqrt{-1}$, $m = 0, 1, 2, \cdots, ip$, in the expression of invariancy for the direct transformation

$$x = X \cos \vartheta - Y \sin \vartheta, \quad y = X \sin \vartheta + Y \cos \vartheta.$$

III. The most general rational integral function of degree i and equal first and second weights q, q in the double system of coefficients

$$\begin{matrix} c_{00} \\ c_{10},\ c_{01} \\ c_{20},\ c_{11},\ c_{02} \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \end{matrix}$$

is directly operated on in such a way as to produce from it the most general gradient of the type i, q, q which has two, Ω_{xz}, Ω_{yz}, of the four annihilators of covariants of ternary quantics

$$\sum_{r=0,\, s=0}^{r+s=p} \frac{p!}{r!\, s!(p - r - s)!} c_{rs} x^r y^s z^{p-r-s},$$

p being any number sufficiently large to allow all the coefficients c_{rs} in the gradient tó be present in the quantic.

KOCH, H. VON: " On regular and singular solutions of certain infinite systems of linear equations."

WHITTAKER, E. T.: " On the functions associated with the elliptic cylinder in harmonic analysis."

This paper is devoted to the "elliptic cylinder functions" defined by the equation

$$\frac{d^2y}{dz^2} + (a + k^2 \cos^2 z)y = 0,$$

where a and k denote constants. This equation presents itself in the theory of linear differential equations as the most natural one to discuss after the hypergeometric equation. Its periodic solutions are shown to be the same as the solutions of the homogeneous integral equation

$$y(z) + \lambda \int_0^{2\pi} e^{k \cos z \cos s}\, y(s) ds = 0.$$

Making this equation the basis of the further developments, the author obtains directly from it the expressions of the elliptic cylinder functions. The one of lowest order (which degenerates into the Bessel's function of zero order when the elliptic cylinder degenerates into a circular cylinder) is

$$ce_0(z) = 1 + \left(\frac{1}{8}k^2 - \frac{7}{2^{13}}k^6 + \cdots\right)\cos 2z + \cdots$$
$$+ \left(\frac{k^{2r}}{2^{4r-1} r!\, r!} - \frac{r(3r+4)k^{2r+4}}{2^{4r+7}(r+1)!\,(r+1)!} + \cdots\right)\cos 2rz.$$

SALTYKOW, N.: "Sur l'intégration des équations aux dérivées partielles."

Let there be a normal system of q equations

(1) $f_i(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n) = 0 \quad (i = 1, 2, \cdots, q)$

solvable with respect to $p_1, p_2, \cdots, p_q$, the corresponding linear system

(2) $(f_i, f) = 0 \qquad (i = 1, 2, \cdots, q),$

admitting $n + \rho (\rho < n - q)$ distinct integrals

(3) $f_1, f_2, \cdots, f_q, f_{q+1}, \cdots, f_{n+\rho}.$

S. Lie has integrated the system (1) by a quadrature when the integrals (3) satisfy a certain condition, which may be replaced by

(4) $dz = \sum_{s=1}^{n} p_s dx_s,$

provided this expression becomes an exact differential in virtue of equations (1) *and the* $n - q + \rho$ *equations obtained by setting the last* $n - q + \rho$ *integrals* (3) *equal to arbitrary constants*

$a_1, a_2, \cdots, a_{n-q+\rho}$. The question then arises whether this restriction diminishes the generality of the solution. It is shown that the integrals obtained from the system (2) are the same on Lie's hypothesis as on that of the author. The latter is sufficient to yield the complete integral of (1) without making use of the complete system of integrals of (2). Moreover it becomes possible to dispense with the functional group. For if we have

$$(5) \qquad z = \varphi(x_1, x_2, \cdots, x_{n-\rho}, a_1, a_2, \cdots, a_{n-q+\rho}) + a$$

$$(6) \quad x_{n-\rho+i} = \varphi_i(x_1, x_2, \cdots, x_{n-\rho}, a_1, a_2, \cdots, a_{n-q+\rho}) \quad (i = 1, 2, \cdots, \rho)$$

and

$$(7) \quad p_s = \psi_s(x_1, x_2, \cdots, x_{n-\rho}, a_1, a_2, \cdots, a_{n-q+\rho}) \quad (s = 1, 2, \cdots, n),$$

it follows that

$$\frac{\partial \varphi}{dx_r} = \psi_r + \sum_{i=1}^{\rho} \psi_{n-\rho+i} \frac{\partial \varphi_i}{\partial x_r}, \quad (r = 1, 2, \cdots, n - \rho).$$

The necessary and sufficient conditions that eliminating the constants $a_{n-q+1}, a_{n-q+2}, \cdots, a_{n-q+\rho}$ *from equations* (5) *and* (6) *furnish the complete integral of* (1) *are given by*

$$(8) \quad \begin{gathered} D\left(\frac{\varphi_1, \varphi_2, \cdots \varphi_\rho}{a_{n-q+1}, a_{n-q+2}, \cdots a_{n-q+\rho}}\right) \gtrless 0 \\ U'_{a_{n-q+i}} \equiv 0 \quad (i = 1, 2, \cdots, \rho), \quad U'_a \gtrless 0, \quad U'_{a_k} \gtrless 0 \\ (k = 1, 2, \cdots, n - q), \end{gathered}$$

where

$$(9) \qquad U'_{a_s} \equiv \frac{\partial \varphi}{\partial a_s} - \sum_{i=1}^{\rho} \psi_{n-\rho+i} \frac{\partial \varphi_i}{\partial a_s}.$$

When these last conditions are satisfied the first n integrals (3) *are in involution.*

Whatever the integrals (3) *may be, it is always possible to satisfy the conditions* (8) *by introducing in formulas* (5)–(7) *new arbitrary constants to denote the initial values of the variables.*

If we have to do with any normal system whatever of partial equations it is unnecessary to seek the complete integration of the corresponding linear system, whenever the integrals obtained satisfy the conditions of the above theorem, since the other integrals may then be obtained either by means of formulas (9) or by aid of Jacobi's theorem generalized.

These results were applied to Imschenetsky's equation

$$F \equiv (x_2p_1 + x_1p_2)x_3 + dp_3(p_1 - p_2) = a,$$

showing how to complete the integration by algebraic elimination.

RÉMOUNDOS, G.: "Sur les singularités des équations différentielles."

When an integral is sought which is to vanish for $x = 0$ we sometimes meet with an interesting singularity characterized by the following properties:

1°. It is possible to obtain from the differential equation a power series that satisfies it formally.

2°. This series is, in general, divergent and consequently there is, in general, no integral which is holomorphic in the neighborhood of $x = 0$ and vanishes at this point. These two properties are consequences of the following, which suffices to characterize the singularity in question: The terms of the differential equation which predominate in computing coefficients of the series by successive differentiation do not have the maximum order as it occurs in the regular cases or the ordinary singularities. Introducing the notion of weight (poids) the characteristic property may be expressed as follows: *No term of maximum order has the maximum weight.* Rules are given in the paper for determining the weights of various terms in the differential equation. This singularity presents itself in equations of the form

$$y = bx + \alpha x^2 y' + F[x, y, y', y'', \cdots, y^{(m)}],$$

where F denotes a function developable in power series in $x, y, y', y'', \cdots, y^{(m)}$ of which the weight is *negative.* If the coefficients α, b, and those of F are all *positive,* there exists no integral of this equation that vanishes at $x = 0$ and is holomorphic in its neighborhood.

HILL, M. J. M.: "The continuation of the hypergeometric series."

The object of this paper is to call attention to certain difficulties which arise in the attempt to apply the *method of ordinary algebraic expansion,* which has been successfully applied to series having one or two points of singularity, to a series with three such points, viz.: the hypergeometric series.

The equation to be proved is

$$F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)=\frac{\pi(\alpha+\beta-\gamma)\pi(-\gamma)}{\pi(\alpha-\gamma)\pi(\beta-\gamma)}F(\alpha, \beta, \gamma, x)$$

$$+\frac{\pi(\gamma-2)\pi(\alpha+\beta-\gamma)}{\pi(\alpha-1)\pi(\beta-1)}x^{1-\gamma}\cdot F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x).$$

The series $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ and $x^{1-\gamma}\cdot F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$ cannot be expanded in series of integral powers of x. Consequently in each series concerned only $n+1$ terms are taken, and an attempt is made to prove that the difference between the two sides tends to zero as n tends to ∞, it being known that $|x|$ and $|1-x|$ are each less than unity. We expand $x^{1-\gamma}$ in powers of $1-x$ and take $n+1$ terms. Then, multiplying by $F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$ it is shown that the terms of degree higher than n can be neglected. The right hand side is thus reduced to a polynomial of degree n in x, and $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ is treated in the same way.

The coefficient of x^r is then transformed into two parts, one of which is the coefficient of x^r in

$$\frac{\pi(\alpha+\beta-\gamma)\pi(-\gamma)}{\pi(\alpha-\gamma)\pi(\beta-\gamma)}F(\alpha, \beta, \gamma, x).$$

The difference between the other part of the coefficient of x^r in $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ and the coefficient of x^r in

$$\frac{\pi(\gamma-2)\pi(\alpha+\beta-\gamma)}{\pi(\alpha-1)\pi(\beta-1)}x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$$

is expressed in the form $k_0(1-x)^n+k_1x(1-x)^{n-1}+k_2x^2(1-x)^{n-2}+\cdots+k_nx^n$ and it is clear that when v is small compared with n the term $k_vx^v(1-x)^{n-v}$ tends to zero as n tends to infinity. But when v is comparable with n further investigation is needed.

CUNNINGHAM, A.: "On Mersenne's numbers."

A Mersenne's number is $M_q=2^q-1$ when q is prime. Taking the 56 primes $q=1, 2, 3, \cdots, 257$, Père Mersenne affirmed (in 1644) that only 12 values of q made M_q prime, viz., $q=1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$; and that the remaining $44 M_q$ were all composite. The grounds for this assertion are not known, and it has up to the present time been only partially verified. Twelve of the M_q have been proved prime, viz., those for which $q = 1, 2, 3, 5, 7, 13, 17, 19, 31, 61$ (not 67), 89 and 127. Twenty-nine have been proved composite and all factors found for eleven of them, viz., those given by $q = 11, 23, 29, 37, 41, 43, 47, 53, 59, 67, 71$; while one or more factors have been found for the eighteen given by $q = 73, 79, 83, 97, 113, 131, 151, 163, 173, 179, 181, 191, 197, 211, 223, 233, 239, 251$. The first factors when $q = 71, 163, 173, 197$ were found by the author.

EVANS, G. C.: "Some general types of functional equations."

BECKH-WIDMANSTETTER, H. A. VON: "Eine neue Randwertaufgabe für das logarithmische Potential."

Hitherto only those boundary value problems have been considered where the function itself or a derivative in a fixed direction is given on the boundary. The author discusses in this paper a boundary condition

$$Ax^2\frac{\partial^2 z}{\partial x^2} + 2Bxy\frac{\partial^2 z}{\partial x \partial y} + Cy^2\frac{\partial^2 z}{\partial y^2} = F(x, y).$$

Applying Fourier's series

$$z = b_0 + \sum_{k=1}^{\infty} r^k(a_k \sin k\varphi + b_k \cos k\varphi)$$

$$F(x, y) = f(\varphi) = d_0 + \sum_{k=1}^{\infty} (c_k \sin k\varphi + d_k \cos k\varphi),$$

the solution is undetermined as to a solution U of

$$Ax^2\frac{\partial^2 U}{\partial x^2} + 2Bxy\frac{\partial^2 U}{\partial x \partial y} + Cy^2\frac{\partial^2 U}{\partial y^2} = 0.$$

By aid of elementary transcendentals U can be expressed linearly in 8 arbitrary constants. Instead of the method of varying parameters, the author has given a direct treatment of the difference equation with constant coefficients, which yields more convenient expressions. He obtains the Fourier series for z and sums it by introducing

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi) \sin k\psi d\psi, \quad \text{etc.}$$

Thus the solution is carried up to the expression as definite integral after the analogy of Poisson's. The author refers, for these complicated expressions, to his own work in the *Monatshefte für Mathematik und Physik,* volume 23 (1912).

PEDDIE, W.: "A mechanism for the solution of an equation of the nth degree."

The principle employed may be seen from a consideration of that system of pulleys in which each of several cords has one end fastened to the body to be raised while its other end carries one of the pulleys of the system. If the body be moved up through a distance a the free end of the last cord descends by the amount $a(2^n - 1)$. Suppose now that the body is immovable while the cords are wound on drums attached to it. If a length a be unwound from the first drum, the free end of the last cord descends through the distance $2^n a$. If a length b be unwound from the second drum, the free end descends by an additional amount $2^{n-1}b$, and so on. Thus, if the length of cord let off the last drum be such that the free end of the last cord has retaken its initial position, we may say that the arrangement solves the equation

$$ax^n + bx^{n-1} = 0$$

in the particular case in which $x = 2$.

If the direction of motion of the drums and the direction of motion of the pulleys are inclined to each other at an angle $\pm\varphi$ when the free end of the last cord has retaken its initial position the corresponding root of the equation is $1 \pm \sin\varphi$.

The cords are replaced by fine steel wires and the action of gravity by the control of wires wound on spring drums. The free end of the last cord is also wound on a spring drum provided with a scale and index. The instrument thus readily gives the value of the function for any value of the variable and conversely.

VOLTERRA, V.: "Sopra equazioni di tipo integrale."

In the absence of Professor Volterra part of his paper was read by Professor Somigliana.

WILKINSON, M. M. U.: "Elliptic and allied functions; suggestions for reform in notation and didactical method."

The paper suggests that a notation based on Weierstrass's sigma formulæ should supersede the sn., cn., dn. notation, which superseded the sinam, cosam, Δam notation many years ago. It also suggests that the σ-function should be defined by the differential equation

$$\sigma\frac{d^4\sigma}{du^4} - 4\frac{d\sigma}{du}\frac{d^3\sigma}{du^3} + 3\left(\frac{d^2\sigma}{du^2}\right)^2 + g_1\left\{\sigma\frac{d^4\sigma}{du^4} - \left(\frac{d\sigma}{du}\right)^2\right\} - \tfrac{1}{2}g_2\sigma^2 = 0.$$

It concluded with the suggestion that more attention than has hitherto been customary should be paid to the differential equation

$$\tau\frac{d^4\tau}{du^4} - 4\frac{d\tau}{du}\frac{d^3\tau}{du^3} + 3\left(\frac{d^2\tau}{du^2}\right)^2 = 0.$$

Throughout the paper was illustrated by examples of formulæ and investigations simplified by the proposed reforms.

ZERVOS, P.: "Sur les équations aux dérivées partielles du premier ordre à quatre variables."

The purpose of this communication was to prove three propositions that are important because of their relation to the calculus of variations.

1°. To every partial differential equation of the first order in three independent variables

$$(1) \qquad F(x_1, x_2, x_3, x_4, p_1, p_2, p_3) = 0$$

corresponds a Monge's equation of the second order

$$(2) \quad f_1(x_1, x_2, x_3, x_4, dx_1, dx_2, dx_3, dx_4, d^2x_1, d^2x_2, d^2x_3, d^2x_4,) = 0.$$

2°. The general solution of equation (2) can be found immediately if we know the complete integral of (1)

$$(3) \qquad V(x_1, x_2, x_3, x_4, a_1, a_2, a_3) = 0.$$

3°. In two communications presented to the Paris Académie des Sciences, April 10 and September 23, 1905, the author gave some general results on Monge equations of the first order from which it follows that to the equation (1) we may assign a Monge's equation of the first order

$$(4) \qquad f(x_1, x_2, x_3, x_4, dx_1\, dx_2, dx_3, dx_4) = 0,$$

equivalent to the equations

$$V = 0, \quad \frac{\Delta V}{\Delta\alpha} = 0, \quad \frac{\Delta^2 V}{\Delta\alpha^2} = 0, \quad \sum_{\kappa=1}^{4}\frac{\partial}{\partial x_\kappa}\left(\frac{\partial V/\partial a_1}{\partial V/\partial a_3}\right)dx_\kappa = 0.$$

Consequently the equations (2) and (4) have in their solutions three equations in common, namely

$$V = 0, \quad \frac{\Delta V}{\Delta \alpha} = 0, \quad \frac{\Delta^2 V}{\Delta \alpha^2} = 0.$$

From this there follow some important geometrical theorems.

RABINOVITCH, G.: "Eindeutigkeit der Zerlegung in Primzahlfaktoren in quadratischen Zahlkörpern."

The paper treats quadratic realms having $D = 1 - 4m$.

Theorem 1.—Given, in a certain realm, two integers α and β of which neither is divisible by the other, if it is possible to find two other integers ξ and η of the same realm satisfying the inequality

$$0 < N(\alpha\xi - \beta\eta) < N\beta,$$

then the class number of the realm is unity.

Definition.—A fraction α / β will be called "disturbing" (störend) when the inequality

$$0 < N\left(\frac{\alpha}{\beta}\xi - \eta\right) < 1$$

admits no solutions in integers.

Theorem 2.—If the class number of a realm is greater than unity, this realm contains disturbing fractions of the form $(p - \vartheta) / q$ where $p < q < m$.

Theorem 3.—The class number of a realm is greater than or equal to one according as the sequence of $m - 1$ numbers $p^2 - p + m$ $(p = 1, 2, \cdots, m - 1)$ does or does not contain composite numbers.

PEEK, J. H.: "On an elementary method of deducing the characteristics of a certain partial differential equation of the second order."

MARTIN, A.: "On powers of numbers whose sum is the same power of some number."

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