Given the family of surfaces $f(x, y, z, v) = 0$, and the curves

$$f = 0, \quad \partial f/\partial v = 0$$

(characteristics); if the equations

$$f = 0, \quad \partial f/\partial v = 0, \quad \partial^2 f/\partial v^2 = 0$$

have a complete solution, corresponding geometrically to a curve, this curve is a contact curve of the series of characteristics, provided

$$\frac{\partial f}{\partial y} \cdot \frac{\partial^2 f}{\partial z \partial v} - \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial y \partial v}, \quad \text{etc.,}$$

are not all zero. It is not necessarily an envelope or an edge of regression. Monge's "arête de rebroussement" is not entirely justified, as is seen in the example of the circles of curvature and spheres of curvature of a space curve.

8. Dr. Velten's paper will appear in full in the next number of the *Jahresbericht.*

9. Professor Voigt first pointed out a number of serious errors in the current mathematical theory of taxes and tariff, and mentioned the remedy found in his recent book: Mathematische Theorie des Tarifwesens (Jena, 1912), namely, by means of a rational construction of the tariff from a mathematical basis. Certain advantages to both the importer and to the government were explained which would ensue from this procedure.

VIRGIL SNYDER.

SHORTER NOTICES.

Differential and Integral Calculus. By Professor L. S. HULBURT. New York, Longmans, Green, and Co., 1912. xviii + 481 pp. with figures.

THE subtitle of this volume, "An introductory course for colleges and engineering schools," indicates the scope of the author's aims. In view of the numerous elementary text books on the calculus, each enjoying more or less popularity at the present time, it might seem a priori that a newcomer in the field would have difficulty in displaying sufficient individuality to warrant its entrance upon the stage. But no reader of Professor Hulburt's book will have any difficulty in recognizing a master hand at all stages from the modest preface to the closing chapter on differential equations. In the matter of notation, of selection and arrangement of topics and of kindred points we may continue to expect a wide variance of opinion, but it is obvious that the author has decided views on these matters, and has not hesitated to incorporate them in his book. It may seem rather presumptuous to criticize without trial what such a successful teacher as Professor Hulburt tells us is the result of test and trial in the class room, but as the fond parent is ever prone to regard the idiosyncrasies and peculiarities of his own children as evidences of promise or genius, the critic may at all times be expected to differ on some points. Many teachers for example prefer the symbol d/dx to D_x, which the present text employs, but both notations are so common that even the beginner should be familiar with both. Typographically the book is well done and quite free from errors, although the very frequent use of a double line within brackets, when stating a theorem or definition consisting of two parts, differing only by the substitution of an alternative word or phrase at one or more places, does not add to the attractiveness nor interest of certain pages. The use of very heavy cancellation marks in the algebraic reduction of illustrative examples is quite conspicuous; but it is obvious that these are mere matters of taste or convenience. The double use, in close proximity, of e as the base of natural logarithms and as the eccentricity of a conic, without explicit statement, might lead to confusion. There seems to be no reason for the author's defining the circular functions as the *inverse* trigonometric functions instead of adhering to the well established meaning as found in various books on the theory of functions as well as on the calculus.

The author has maintained clearness without sacrificing rigor, wherever he has essayed a proof. But he has not hesitated to use a theorem without proof where he believes the rigorous proof would not appeal to the ordinary beginning student. As a typical illustration of this we might mention the theorem, "In taking a mixed partial derivative of any order, provided the function be continuous, it is immaterial in what order the differentiations are performed." Among others the general theorems on limits, and the ratio test for the convergence of infinite series are stated without proof.

This omission of proofs which do not enlighten is in accordance with generally accepted sound pedagogy in other courses. Again the author does not hesitate to introduce a new idea by a working definition or statement of a theorem, reserving the rigorous analytic proof for later pages. For example, he begins by defining a function as being discontinuous for real values of the variable "wherever it has a break in its graph." The analytic definition of discontinuity follows later. The author uses the method of limits, as distinguished from the method of rates, and appeals strongly to the geometrical intuition from the start. But at no time does he seem to lose sight of the important truth that a clear conception of the fundamental ideas of the calculus is of more importance than the mere ability to manipulate the formal side, even in applications. For purposes of convenience the volume is divided into six books, each of which is more or less of a unit in itself. The first two constitute a quite complete elementary course in themselves.

Book I, with its seventeen chapters and 175 pages, comprises considerably more than one third of the text and is a complete elementary course in the differential calculus of functions of one variable. The introductory chapters on functions, discontinuities of functions, and limits are models of clearness and should enable any intelligent beginner to start upon his course with an adequate understanding of these rather difficult concepts. The distinction between the *equation* of a graph, and the *function* which the graph represents is made extremely clear. In this book the study of the convexity and concavity of algebraic curves, and of their maxima, minima, and flexes* is taken up before the formulas for differentiating logarithmic and exponential functions are derived. The illustrations and examples in this book are almost entirely geometrical, except in the brief treatment of velocity. Chapters X to XVI are devoted to plane curves, separate chapters being devoted to parametric equations, to polar equations, and to cycloidal curves. Curvature, involutes, and evolutes are treated briefly. The collection of curves is a very interesting one. It may be noted that, despite the amount of space devoted to curves, the subjects of asymptotes

* Professor Hulburt adopts the terms "flex," "flex tangent," etc., instead of the longer "point of inflection," "inflectional tangent," etc. The terms are due to Professor Frank Morley.

and of general methods of finding singularities are not taken up. As these topics throw little added light on the meaning or application of derivatives, the omission is doubtless warranted, and they are rightly reserved for works on higher plane curves or projective geometry, where they may be completely treated. However interesting these chapters are to the mathematical student, it may seem to some teachers that they might be curtailed or, preferably, be reserved in part for some later book, although logically they belong here. There is a demand from the physicists and others in engineering schools that the formal elements of both differential and integral calculus be given to the student as early as possible, and this goal might be approached by deferring these chapters of Book I until after Book II had been studied. Indeterminate forms are concisely treated in the last chapter.

Book II covers 90 pages, of which but 25 are devoted to the technique of formal integration, the major stress being on integration by the aid of tables. Of the ordinary special methods of integration, but three are explained in detail—algebraic substitution, trigonometric substitution, and integration by parts. The reduction formulas have been deservedly ejected from their time honored seat, and do not appear in the volume at all, while the integration of rational fractions is reserved for a chapter in a later book. Nineteen pages are devoted to the applications of integration in kinematics, so that this topic is treated with some detail, and accomplishes well its obvious purpose of fixing the significance of the constants of integration and of familiarizing the student with typical applications of the indefinite integral. The remainder of this book is devoted to the simple applications of the definite integral,—calculating the lengths of arcs of plane curves, the areas under them, and the surfaces and volumes of solids of revolution. The clearness of statement in the two chapters on the definite integral deserves special note as the student is prone to swallow this part of the theory on faith and to trust to his works in solving examples to bring about his salvation. Yet at the very beginning the author draws a conclusion after what seems to be the least convincing argument in the entire book. After showing that x^3 and $x^3 + c$ are both integrals of $3x^2$, he immediately states, "Hence the *most general* (italics mine) integral of $3x^2$ is $x^3 + c$." It would seem to the reviewer that a brief table of integrals might well have been included in the volume as a matter of convenience if not of necessity, since the author specifically aims to teach the students to use a table skilfully. The references to paragraphs in an alien book are at times slightly confusing, and it seems, in general, unwise to refer to paragraph numbers which may be changed at the whim of another. Many teachers will regret the absence of a discussion of methods of approximate integration.

Book III, which is devoted to an introduction to analytic geometry of three dimensions, will be warmly welcomed by teachers who find it necessary to take up the calculus without a formal course in solid analytics. The essentials, everything required in the applications of the calculus, are given compactly (some will say too compactly) but clearly and comprehensively in the compass of 20 pages. The teacher whose students have covered this subject matter will find it convenient for review and reference, whereas others will probably find no serious break in continuity since the necessity for a three-dimensional coordinate system is presented by the nature of the problems to be attacked. Volumes of simpler solids and areas of surfaces are treated in this book by single integration only.

Book IV, of 55 pages, is a brief calculus of functions of several real variables. Partial and total derivatives with their geometrical interpretation and application, and multiple integrals with their application to areas, volumes, surfaces, and centers of mass are the principal topics treated. The theory of partial and total derivatives and differentials is treated in considerable detail and one sees a trace of humor in the closing remark, "We repeat . . . that small advantage accrues from the use of differentials. They are relics of the early days of the calculus, and *as the reader is probably now ready to admit* (italics mine) constitute for the beginner an obstacle to the understanding of the calculus rather than an aid. Nevertheless they are in almost universal use and it is therefore necessary that the student of mathematics become thoroughly familiar with them." Teachers may differ as to the utility of this relic. Such topics as work, fluid pressure, and center of pressure are omitted. It is doubtless true that these subjects strengthen very slightly the grasp of the student on the principles and methods of the calculus, as the mathematics is lost sight of in struggling with the ideas of physics and mechanics involved. The moment of inertia integral is also not introduced at all, and center of mass is defined as an integral, without any physical interpretation or derivation.

Book V, of 72 pages, is devoted to a few special topics, chiefly to Taylor's and MacLaurin's theorems, De Moivre's theorem, hyperbolic functions, integration of rational fractions, and envelopes. Many teachers doubtless prefer to take up some of these topics earlier, to which the author would certainly not object, as they are collected in this rather heterogeneous book for convenience of omission or inclusion. The devotion of a few pages to such an important formula as De Moivre's theorem seems justified, although it is omitted from most texts in common use. The treatment of Taylor's theorem possesses peculiar clearness and charm. Based primarily on the extension of the law of the mean, the development is first in the finite form where the remainder is retained and discussed, and then in the infinite form, applying of course only when the series is convergent,—a treatment which certainly tends to clearness. The caustic is a leading example of an envelope.

Book VI is entitled "An Introduction to Ordinary Differential Equations," and the subject is admirably treated within the limits of 30 pages. It is vastly more than a mere tabulation of methods of solving certain types of differential equations. A half dozen pages are devoted to clearing up the ideas involved in such terms as "solution of a differential equation," "particular integral," "complete primitive," and the like. Then all classes of equations ordinarily met by students in applied mathematics are treated, with illustrative examples. Clairaut's equation is given largely as a curiosity. It would seem to the reviewer that it would have been worth while to explain in somewhat more detail the special method of solving linear differential equations of the first degree when the right hand member contains a term which is also a term of the complementary function, as it would throw added light on the whole topic. The remarks by the author on this subject (page 446) are scarcely as full as the student will need to make them truly enlightening.

Professor Hulburt has, as is clear from the above outline, attacked the problem of producing a text book suitable for both colleges and engineering schools with characteristic vigor. Whether the course be one to finish off the mathematical studies of the college student, or to mark the completion of formal mathematics for the prospective engineer, or be but the stepping stone to further advanced study for the mathematical student, the arrangement and the presentation in this volume will materially aid the teacher and student. There is sufficient of the spirit of research and rigorous analysis to meet the demands of the last class, while the necessities of the others have not been neglected. The lists of problems are of sufficient variety and extent to meet all ordinary requirements. It will be quite clear to the careful reader that both in illustrations and in exercises, the author has avoided those examples which, however interesting in themselves, present their greatest difficulty or interest because of the dynamics, physics, or other science involved, and beyond that shed little or no new light on the methods of the calculus. In fact the field of dynamics is entirely avoided. On the other hand the author does not attempt to conceal or minimize the real difficulties of calculus by employing trivial examples. The answers to all exercises are conveniently assembled at the end of the text. The index is exceptionally complete. There is no doubt but that this text will be a valued addition to the teacher's library and will find a deserved admission into many class rooms.

D. D. LEIB.

Theorie der Zahlenreihen und der Reihengleichungen. By ANDREAS VOIGT. Leipzig, Göschen, 1911. viii + 133 pp.

THE two fundamental ideas which underlie this work are the following:

1. Instead of considering a number as isolated, one may think of it as belonging to a sequence. Thus the question as to whether b is divisible by a is the question as to whether b belongs to the sequence $\cdots, -2a, -a, 0, a, 2a, \cdots$.

2. Instead of expressing an integer N as a polynomial in x of the form

$$N = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

one may write it in either of the forms

$$N = b_0x(x-1)\cdots(x-n+1) + b_1x(x-1)\cdots(x-n+2) + \cdots + b_{n-1}x + b_n,$$

$$N = c_0x(x+1)\cdots(x+n-1) + c_1x(x+1)\cdots(x+n-2) + \cdots + c_{n-1}x + c_n.$$