LET US HAVE OUR CALCULUS EARLY.

The Calculus for Beginners. By J. W. MERCER. Cambridge (England), University Press, 1910. xiv + 440 pp.

THE decline of the Græco-Roman empire over our collegiate studies has been most marked and is apparently extending almost to extinction. Many reasons may be assigned for this fall. One is the widening range of knowledge and interest, another the passing of the professional and utilitarian aspects of Greek and Latin, another the failure of these subjects to make good.

We must not forget that our colleges were for the most part started by ministers of the gospel, and chiefly as divinity schools. In the early days, even if not to such an extent at present, Greek and Latin were just as utilitarian and necessary professional subjects for the young student of divinity as mathematics now is for the electrical engineer and the physicist. The talk of their value as cultural and disciplinary studies is probably of later and comparatively recent date, an invention of those vast vested interests who would delay, even though they cannot stay, the march of progress away from them. The percentage of our present undergraduate collegians for whom the ancient languages are utilitarian and professional is very small.

That culture and valuable intellectual discipline are best obtained by application to subjects which are neither useful nor interesting to the student, and over which he never obtains even a mediocre mastery, is an idea which is losing ground and must necessarily lose ground despite the extent and intrenchment of the aforesaid vested interests. The fact is that Greek and Latin do not make good. After studying them for six to eight years the student cannot read them with ease or profit; they form but a comparatively small part of his mental outfit, whether for pleasure or work.

If the classics were still of widespread professional necessity, or if the study of them throughout the preparatory course and for two or three years in college gave a pleasurable mastery over them, it is far from likely that they would occupy their present low estate even in the face of the widening range of knowledge, the transfer of executive interest in our colleges from accomplishment to equipment and from high standards to numbers, and the accompanying transition of student interest from study to sport.

We mathematicians, however, are in no position to gaze upon the motes in the eyes of our classical brethren, to whom we can hardly compare ourselves favorably. For there has been a great decline in the sway of mathematics over collegiate education. We cannot attribute this decline nearly so much as in the case of classics to a passing of professional or utilitarian interest. Fifty years ago who needed mathematics? Those who were to teach it, a few theoretical astronomers, some terrestrial surveyors. Now, added to these, there is the vast host of engineers, civil, mechanical, electrical, and others, who should be familiar with the calculus. We cannot so well attribute the decline to the widening range of knowledge; for to some of the newer subjects, notably physics, mathematics is complementary rather than competitive. We suffer, of course, in common with all the severer disciplines, by the introduction of large numbers of "cultural" and "snap" courses, by the presence in college of great numbers of fellows neither primarily nor seriously there for the sake of intellectual advancement. But our chief difficulty is that we do not make good.

By not making good is meant that after a long course in preparatory mathematics and a couple of years in college most students, indeed all except the most extraordinary, are unable to use their mathematics, that is, they find their mathematics, especially that learned in college, an ineffective part of their mental outfit whether for pleasure or work. Were it not for a fancied unique training, more imaginary than real, that mathematics is supposed to give, were it not for the obvious professional necessities of the engineer, we should find collegiate mathematics worse off than classics, and most of us mathematicians without a job. With a full realization of the fact that our students cannot get out of our courses as much as we may wish them to get, with all proper disdain for those practical persons who desire the student to acquire mathematics merely as a tool—an impossibility—, those of us who are honest and not too conceited cannot fail to feel that we do not make good. As external evidence of this fact we need only consider the large number of texts (by incompetent mathematicians) now appearing under such titles as Mathematics for the Chemist, Engineering Mathematics, and so on.

One of the main troubles with us is that we do not select the right subjects to teach in the early collegiate years.* There is no sense in giving the freshman a considerable course on advanced algebra. The subject is abstract and deals with topics and ideas relatively unimportant for the student. Yet advanced algebra is often taught as a prerequisite to calculus. It is unfortunate to force the freshman through an extensive course in analytic geometry. So much of the subject as deals with rectangular (and possibly polar) coordinates and with the tracing of simple curves is indeed necessary, and is closely related to methods in use in a wide variety of everyday studies. This much we may call graphical representation. But the real essence of analytic geometry, the obtaining of geometric results by algebraic means, is of small use save to the pure mathematician and is rarely comprehended by the immature student.

That mathematical subject which is most vitally important for the general student is the calculus, differential and integral. It is no less important for the infrequent future mathematician. It is the calculus which should form the greatest possible portion of the ordinary freshman course in mathematics. When the calculus is postponed until the sophomore year and follows somewhat elaborate courses on trigonometry, advanced algebra, or analytic geometry, it can be made so abstract, so formal, so purely mathematical as to leave little impression upon any save the best students. This is the chief danger in postponing it. If the teacher can rely upon only a very meager preparation, he has to keep the analysis elementary to the point where it offers small difficulty even to the student who is not particularly facile with his algebra, he is able to give his time to a variety of simple but fundamental applications which rivet the attention of the student, he has a chance, which he otherwise does not have, to make good.

Such considerations as these are probably at the back of the

* In an engineering school where the student is required to take mathematics for a definite period the order and treatment of topics does not need to be so tempered to the shorn lamb as in a college where there is a tolerably free range of election, where many more will take one year of mathematics than two, and where the conduct of the first year is therefore of prime importance both for the general good and for subsequent elections in mathematics.

somewhat general, and the highly commendable, tendency which we are now witnessing toward the earlier introduction of the calculus. This is undoubtedly the idea which has inspired The Calculus for Beginners by J. W. Mercer, head of the mathematical department at the Royal Naval College, Dartmouth, England. The author states in his preface: "I have been guided by my conviction that it is much more important for the beginner to understand clearly what the processes of the calculus mean, and what it can do for him, than to acquire facility in performing its operations or a wide acquaintance with them. I had much rather that a boy, confronted with a problem, should, after analyzing it, be able to say 'If I could differentiate (or integrate) this function of x, I could solve the problem,' than that he should be able to perform the operation without seeing its bearing on the problem. . . . A boy is supposed to know his elementary algebra and trigonometry and to have some slight acquaintance with the coordinate geometry of the straight line. He should be able to write down the equation of a line through a given point with a given gradient and should know the relation between the gradients of perpendicular lines. He is also expected to have had some practice in drawing graphs from their equations and to know what these graphs mean. . . . The subject has been taught here during the last few years to boys of 16 on the lines of this book. . . ."

Mercer must be a careful teacher; he explains everything very clearly and with great detail, indeed some would say with too great detail. There is no reason why he or any one else should fail in teaching this book to boys of sixteen. Except for the definition of the tangent of an angle no trigonometry is required before page 261, which is two thirds of the way through the book (exclusive of answers and index). To this point the analysis involves merely polynomials in powers (not necessarily integral) of the variable; yet all the primary ideas of differential and integral calculus have been presented, and all the usual applications to maxima and minima, areas, volumes, moments of inertia, and the like, have been given. The requisite amount of coordinate geometry could be given in ten lessons surely, and probably could be explained incidentally to the study of the text without even those preliminary ten lessons.

The exercises set for the student are exceedingy varied and numerous.* They are distributed a few at a time throughout the text as each new principle is explained. An especially valuable feature, however, is found in the long collections of miscellaneous exercises after Chapters VI, VIII, and XI, which allow the student to be tried out in review upon the chief sections of work, namely, the differential calculus of x^n, the integral calculus of x^n $(n \neq -1)$, and the differential and integral calculus of the other elementary functions. It would be safe to say that the student who has been led carefully through the first 260 pages of the text, although being restricted to using only polynomials in x^n, would have a better grasp of the calculus, whether for pleasure or or profit, than he could acquire from the ordinary run of texts on calculus. This ground could probably be covered convincingly in about sixty lessons, and with freshmen.

Chapter I treats uniform speed, average speed, speed at an instant, rate of increase, function, uniform gradient, average gradient, gradient at a point, tangent to a curve, rate of increase as a gradient, test of approximate equality, ratio of continually diminishing quantities. The last two sections give a very clear idea of infinitesimals without the introduction of technical terms. The second chapter, called differentiation from first principles, treats of ds/dt, its connection with $\Delta s/\Delta t$, of dy/dx as a rate of increase, as the ratio of time rates of increase of x and of y, as a gradient, of the significance of the sign of dy/dx. Observe that the author uses speed or rate as primary concept instead of gradient. This is a good thing. Speed and rate are more concrete than slope. The student should learn to think in physical terms, to consider a curve as a convenience in representing physical phenomena, not as an end in itself. The great trouble with students who know something about curves and analytic geometry is that they try to use too much of this knowledge. To them a circular plate suggests coordinate axes and the equation $x^2 + y^2 = a^2$, whereas in most problems in the application of calculus these are entirely superfluous.

After these preliminaries, which fill 74 pages, we come to the differentiation of x^n, to maxima and minima, to small errors and approximations, and to the inverse of differentiation, all of which occupies 78 pages, and is followed by a collection of 50

* The exercises have also been reprinted and published separately from the text.

review exercises. It is interesting to observe that the author treats some problems in maxima and minima where he has to use such a device as finding the maximum or minimum of the square or reciprocal of the given function, because the differentiation would otherwise be impossible (unless obtained from first principles) with the formal methods thus far developed. Some would consider this unfortunate. But it seems to us one of the most fortunate things that can arise. Ingenuity is a very desirable trait to cultivate. The advantage of studying calculus with small preparation and with restricted formal development is precisely that there is wide play for that sort of ingenuity which the student who will later use his calculus is most likely to need. As a matter of fact unless an engineer can squeeze out a solution of his problem by simple analysis alone, he is not likely to trouble to obtain a solution himself.*

Chapter VII is on the integral as a limit of a sum, areas of curves, mean or average ordinate. Simpson's rule is explained and applied. The author nowhere shuns a reasonable amount of calculation, and this is one of the elements that makes his work concrete, practical, and full of meaning. In the eighth chapter further applications are taken up: finding the distance when the speed is given, calculating the work done by a force, finding fluid pressures, volumes of revolution, moments of inertia, centers of gravity, and centers of pressure. Here follows another set of 50 miscellaneous exercises.

The title of Chapter IX is the differentiation of trigonometrical ratios, but we observe that the integration of simple trigonometric expressions is taken up simultaneously. This is as it should be. The exercises are well chosen so far as they go, but it would have been better if they had been more numerous and decidedly more varied. The chapter on polar coordinates (the fourteenth and last in the book) might well have been introduced at this point, furnished with some really satisfactory exercises, and elaborated to include the calculation of moments of inertia, centers of gravity, and the like, of such important figures as yield more easily to treatment by polar

* The idea that an engineer should have in his course only this minimum simple analysis which he is likely himself to use is entirely mistaken; he should have far more. Indeed any student who has such an excellent freshman course as the first eight chapters of this book would make, should then have a systematic drill in formal analysis, including differential equations, with varied applications among which should be found considerable mechanics. This would leave him at the end of his second collegiate year with a very effective knowledge of mathematics.

than by rectangular coordinates. It would have been an excellent review and elaboration upon all the previous ideas in the text.

In Chapter X we find the differentiation of the product, quotient, and function of a function, of inverse and of implicit functions. This is pure formal differentiation, with a trifle of integration, and relatively few applications. In the following chapter we come to the differentiation of n^x, and a general treatment of the exponential and logarithmic function, including simple integrations, the compound interest law, and simple differential equations. The treatment of e is excellent. The author computes with the aid of tables the derivatives of 2^x, 3^x, 2.5^x, and concludes that there must be a number e such that if $y = e^x$, then $dy/dx = y$. The student checks this by differentiating with the tables the function 2.7183^x. The chapter is followed by a collection of 100 miscellaneous exercises.

Chapter XII contains the approximate solution of equations (Newton's method). In Chapter XIII methods of integration by substitution and by parts are given, and the work closes (except for answers and index) with the previously mentioned unsatisfactory chapter on polar coordinates.

We consider that the author would have improved his book had he placed the chapter on the exponential and logarithmic function before Chapter X on the rules of formal differentiation; indeed it might have been still better to have placed this chapter before that on the trigonometric ratios, and to have given more attention to problems requiring statement in a simple differential equation integrable by logarithmic or exponential functions. Such equations arise very frequently in the applications. This, however, is a small matter.

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