34. This paper is a more general treatment of the subject presented to the Society by Professor Bates in April, 1910. In the former paper, only one value of λ (namely, $n - 1$) was considered.

H. E. SLAUGHT,
Secretary of the Chicago Section.

SHORTER NOTICES.

Leçons sur le Prolongement analytique. By LUDOVIC ZORETTI. Paris, Gauthier-Villars, 1911. vi+115 pp.

IT is the plan of this little volume, like that of its fellows of Borel's excellent collection of monographs on the theory of functions, to conduct a student from a presupposed minimum of knowledge—but hardly of capacity—to the actual present frontier of science. This is evidenced by the many questions raised, but not to-day answerable, to be found scattered through its pages. It consists, in substance, of lectures delivered by its author at the Collège de France during the year 1908–1909; while it may be said to have for its subject such questions of the theory of functions as can be best treated, not by the methods of Cauchy and Riemann, but by going back to Weierstrass's fundamental definition of an analytic function. It is in this sense that the title is to be understood. Weierstrass's definition gives rise, in the general case, to a function with an infinite number of branches. The small advance that has been made in the study of such functions, the importance of which is incontestable (in the analytic theory of differential equations, for instance), is due, in part, to its difficulty, in part to the lack of appropriate tools. In particular—it is our author's idea—progress in the theory of these functions depends on progress in the theory of the Riemann's surface of an infinite number of sheets (as this latter is conceived of by Poincaré in his paper on the uniformization of functions, *Acta Mathematica,* volume 31, 1907). "The object of this book will have been accomplished if I succeed in persuading the reader of the great interest attaching to investigations of these surfaces" (page 43).

The work begins, after an interesting and suggestive introduction of six pages, with a chapter of 22 pages on (closed) sets of points in one and two dimensions. The object here was to develop for use later certain properties which could not be considered to form part of the "classic" theory of the subject.

Chapter II, on the general idea of an analytic function, occupies more than a third of the book. A start being made from Weierstrass's definitions of analytic continuation and analytic function, some of the complexity of these notions is brought out by examples; as, for instance, that a point may be a regular point for each branch of a function, yet itself be a point of condensation of singular points.—What kind of set do the singular points of a function constitute? The answer to this question is, in the case of a one-valued function, that it may be any given (closed) set. Similarly a many-valued function exists having for its singular points any given enumerable infinity of closed sets; but whether this is the most general case is not known (see, however, the note on page 51). In attempting to extend to many-valued functions the idea of a natural boundary, we are met by the fact that a function may have a singular line which is not a natural boundary for any of its branches: this is exhibited by an example in the note at the end of the volume. A function may even have a limited domain of existence although no one of its branches has a natural boundary; this phenomenon our author expresses himself as "forced to believe in, but unable to explain." A distinction has then to be made between a natural boundary for a branch of a function and one for the function itself.

Singular points may be classified, not only by the kind of set they belong to, but secondly by the behavior of the function there. This introduces us to Painlevé's notion of the domain of indetermination of such a point.* The domain of indetermination of a point viewed as a singular point of a given branch of a function may be roughly described as the set of values of the function that may be reached by analytic continuation from the given branch without leaving an infinitesimal neighborhood of the point. Then transcendental singularities may be classified as follows: *ordinary* if the domain of indetermination be a point ($z = 0$ for $\log z$), otherwise *essential*; the domain of indetermination in this latter case being either *complete*, where it comprises the whole plane (the isolated essentially singular point of a one-valued function), or else *incomplete*.

* *Comptes rendus*, vol. 131 (1900).

Chapter III takes up Weierstrass's theorem as to the behavior of a one-valued function in the neighborhood of an (isolated) essential singularity—with a proof by the methods, based on analytic continuation, used in this book—and Picard's theorem.

Can Weierstrass's theorem be extended to other kinds of singular points (of a one-valued function)? This question is treated in Chapter IV. Intermediate between the isolated singular point and the singular line comes a point "belonging to a discontinuous set" (phrase exactly defined pages 54, 16). For certain of these cases, too, Weierstrass's theorem holds. But the attempt at a proof in the general case (sketched by Painlevé in his Stockholm lectures) failed; while the proof given by our author in his thesis that such a point was, at any rate, an essential transcendent singularity has turned out to be unsound. It now appears that, in point of fact, not even this narrower proposition is true; for Denjoy has produced an example of a function whose singular points are all of the kind in question, that is, each belongs to a discontinuous set, while the function, nevertheless, remains continuous throughout the plane; so that these transcendental singularities are all merely ordinary ones.

For one-valued functions with singular lines general results are almost non-existent. Chapter V deals with such functions and raises questions like the following. It is known that a function continuous in a region cannot have an *isolated* singular line there: can we say more, that a function with nothing beyond singularities where it remains continuous (ordinary transcendent singularities) is necessarily holomorphic?

Chapter VI concerns itself with an important class of many-valued functions, namely, the inverses of one-valued, and, in particular, of integral functions. The singular points of such a function can be at most of the ordinary transcendent variety. A property pretty much characteristic of them is that at least one of the areas in the plane of the function corresponding to a small circle, in the plane of the independent variable, about the singular point includes the point at infinity.

The reviewer might mention a couple of unimportant slips that have come under his notice: page 1, note 2, the reference should be to the *Monatsberichte;* page 43, note 2, the title of Boutroux's book is incorrectly given.

FRANK IRWIN.