so that from the result of Kronecker's discussion there follows, for numerical equations, a proof of the fundamental theorem of algebra as ordinarily stated.

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# TWO CONVERGENCY PROOFS.

BY PROFESSOR ARNOLD EMCH.

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## 1. *Introduction.*

IN the study of automorphic functions defined within a fundamental domain $G$ formed by two non-intersecting circles in the "elliptic" case and by two tangent circles in the "trigonometric" case, it is necessary to prove the convergence of certain fundamental series, as has been shown by Schottky recently.* In the first case the substitutions of the cyclic group associated with it may be written in the form

$$(1) \qquad \frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x - a}{x - b},$$

where $\lambda$ may assume all integral real values between $-\infty$ and $+\infty$ and where $a$ and $b$ are invariant in all substitutions of the group. When $x$ describes the circle $K_{-1}$ of the fundamental region $G$, $x_1$ describes the circle $K_1$ forming the other boundary of $G$. $x_\lambda$ describes a circle $K_\lambda$ and all circles $K_\lambda$ of the pencil with $a$ and $b$ as limiting points divide the whole $x$-plane into a set of regions corresponding to the substitutions of the group.† Schottky bases his proof of the convergence of the series $\Sigma r_\lambda$ of the radii of the circles $K_\lambda$ on the invariance of the expression

$$\frac{(r_\lambda^2 + r_{\lambda+1}^2 - e_\lambda^2)^2}{4r_\lambda^2 r_{\lambda+1}^2} - 1.$$

* "Ueber die Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) = F(x)$ genügt," *Journal für reine und angewandte Mathematik*, vol. 143, pp. 1–24 (May, 1913).

† See also Klein-Fricke, Vorlesungen über die Theorie der elliptischen Modulfunctionen, vol. 1, pp. 163–207.

This is a positive quantity and is equal to $-\sin^2\phi$, where $\phi$ is the imaginary angle of intersection of $K_{-1}$ and $K_1$ and appears also in the invariant distance

$$|a-b| = \frac{2r_\lambda r_{\lambda+1}}{e_\lambda}\sqrt{\frac{(r_\lambda^2+r_{\lambda+1}^2-e_\lambda^2)^2}{4r_\lambda^2 r_{\lambda+1}^2}-1}.$$

In the second case the substitutions may be written in the form

$$\frac{A}{x_\lambda - a} = \frac{A}{x-a} + \lambda. \tag{2}$$

The domains corresponding to the substitutions of the group are bounded by a parabolic pencil of circles with the common point of tangency $a$. Taking for $X$ and $X'$ any two points of the $x$-plane different from $a$, Schottky proves the absolute convergence of the series

$$\sum_{\lambda=-\infty}^{+\infty}(X_\lambda - X_\lambda') \tag{3}$$

by first proving that of $\Sigma(x_\lambda - x_\lambda')$, in which $x$ is any point different from $a$, and $x'$ a point between $x$ and $x_1$ on the circle through $a$, $x$ and $x_1$. For both the series $\Sigma r_\lambda$ and (3) I shall give direct proofs.

## 2. *Convergence of* $\Sigma r_\lambda$.

In the substitution

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda\frac{x-a}{x-b} \qquad (|q| \neq 1),$$

assume first $|q| = \rho > 1$ and let $x$ describe the circle $K_{-1}$ around $a$, so that

$$\left|\frac{x-a}{x-b}\right| = \sigma < 1.$$

Assuming furthermore $\rho\sigma > 1$, then $x_1$ describes the circle $K_1$ about $b$, so that the fundamental domain $G$ bounded by $K_{-1}$ and $K_1$ extends to infinity. All circles $K_\lambda$ with $\lambda = 1, 2, 3, \cdots$ enclose $b$; all circles $K_\lambda$ with $\lambda = 0, -1, -2, -3, \cdots$ enclose $a$. Designating the distance of the centers of $K_{-1}$ and $K_1$ by $e$, the radius of $K_\lambda$ for positive values of $\lambda$ is easily found from

$$\left|\frac{x_\lambda - a}{x_\lambda - b}\right| = \sigma\rho^\lambda$$

as

(4) $$r_\lambda = \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1},$$

and in case $\lambda \leqq 0$ as

(5) $$r_\lambda = \frac{e\sigma\rho^\lambda}{1 - \sigma^2\rho^{2\lambda}}.$$

To prove first the convergence of

(6) $$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1},$$

we write

$$r_\lambda = \frac{e}{\sigma} \frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^2}.$$

Now, since $\sigma < 1$, $1/\sigma^{2\lambda} > 1/\sigma^2$, and from $\sigma\rho > 1$, $\rho > 1/\sigma$, we have

$$\rho^{2\lambda} - \frac{1}{\sigma^{2\lambda}} > 0$$

and

$$\rho^{2\lambda} - \frac{1}{\sigma^{2\lambda}} < \rho^{2\lambda} - \frac{1}{\sigma^2}.$$

Consequently

$$\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^2} < \frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^{2\lambda}}.$$

But

$$\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^{2\lambda}} = \frac{\rho^\lambda}{(\rho^2 - 1/\sigma^2)(\rho^{2(\lambda-1)} + \rho^{2(\lambda-2)} \cdot 1/\sigma^2 + \cdots)}$$

$$< \frac{\rho^\lambda}{(\rho^2 - 1\sigma^2)/\rho^{2(\lambda-1)}},$$

so that finally

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1} < \frac{e}{\sigma(\rho^2 - 1/\sigma^2)} \sum_{\lambda=1}^{\infty} \frac{1}{\rho^{\lambda-2}},$$

or

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1} < \frac{e\rho^3}{\sigma(\rho - 1)(\rho^2 - 1/\sigma^2)},$$

i. e., less than a finite positive quantity. For negative values of $\lambda$, $\lambda = -\mu$ ($\mu \geqq 0$), we have the series

$$\sum_{\mu=0}^{\infty} r_{-\mu} = \sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2}.$$

As $\sigma^2 < 1$,

$$\frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2} < \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - 1}.$$

But

$$\rho^{2\mu} - 1 = (\rho^2 - 1)(\rho^{2(\lambda-1)} + \rho^{2(\lambda-2)} + \cdots) > (\rho^2 - 1)\rho^{2(\lambda-1)},$$

so that

$$\sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2} < \sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{(\rho^2 - 1)\rho^{2(\lambda-1)}},$$

or finally

$$\sum_{\mu=0}^{\infty} r_{\mu} < \frac{e\sigma\rho^3}{(\rho - 1)(\rho^2 - 1)},$$

less than a positive finite quantity.

Consequently

$$\sum_{\lambda=-\infty}^{+\infty} r_{\lambda} = \sum_{\lambda=1}^{+\infty} r_{\lambda} + \sum_{\mu=0}^{+\infty} r_{\mu}$$

is convergent. In a similar manner the convergence can be proved for the sum of the radii of any other loxodromic cyclic group.*

3. *Convergence of* $\Sigma(x_{\lambda} - x_{\lambda}')$ *for a Parabolic Cyclic Group.*

In

$$\frac{A}{x_{\lambda} - a} = \frac{A}{x - a} + \lambda, \quad \text{put} \quad \frac{A}{x - a} + c = z,$$

where $c$ is any constant. Putting $z - c = \zeta$, then by means of this transformation the $x$-plane is transformed into the $\zeta$-plane, and we find

$$\zeta_{\lambda} = \zeta + \lambda$$

* Schottky proved the convergence of the sum of radii for a more general set of circles in an article: "Ueber eine spezielle Function, welche bei einer bestimmten linearen Transformation ihres Argumentes unverändert bleibt," *Journal für reine und angewandte Mathematik,* vol. 101, pp. 231–236 (1887).

and

$$x_\lambda - a = \frac{A}{\zeta + \lambda}, \quad x_\lambda' - a = \frac{A}{\zeta' + \lambda}.$$

Consequently, for $\zeta \neq \zeta'$ and not equal to real integers,

$$(7) \quad \sum_{\lambda=-\infty}^{+\infty} (x_\lambda - x_\lambda') = A(\zeta' - \zeta) \sum_{\lambda=-\infty}^{+\infty} \frac{1}{\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'}.$$

The absolute value of $[\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta']$ has the form $+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta}$, where $\alpha$, $\beta$, $\gamma$, $\delta$ are real numbers. We can write

$$+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta}$$

$$= + \lambda^2 \sqrt{1 + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2} + \frac{\gamma}{\lambda^3} + \frac{\delta}{\lambda^4}} = + \lambda^2 \sqrt{1 + \phi(\lambda)}.$$

Assuming an arbitrarily small positive quantity $\epsilon$, it is always possible to choose a positive number $n$ large enough so that for all values of $|\lambda| > n$, $|\phi(\lambda)| < \epsilon$. If, for such values of $\lambda > n$, $\phi(\lambda) > 0$, then

$$+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta} > \lambda^2$$

and

$$\sum_{\lambda > n}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} < \sum_{\lambda > n}^{+\infty} \frac{1}{\lambda^2},$$

for all values of $\zeta$ and $\zeta'$ different from a real integer. If, for the same values of $\lambda$, $\phi(\lambda) < 0$, then

$$\lambda^2 \sqrt{1 + \phi(\lambda)} > \lambda^2 \sqrt{1 - \epsilon},$$

and

$$\sum_{\lambda > n}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} \lesssim \frac{1}{\sqrt{1 - \epsilon}} \sum_{\lambda > n}^{+\infty} \frac{1}{\lambda^2}.$$

For negative values of $\lambda$ we have evidently in the first case $\phi(\lambda) < 0$, in the second case $\phi(\lambda) > 0$, so that the two cases for negative values of $\lambda$ lead again to convergent series. Hence it is always possible to determine a number $n$ large enough so that for all values $\lambda > n$, and all values $\zeta$, $\zeta'$ different from an integer, the series

$$\sum_{\lambda}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} + \sum_{-\lambda}^{-\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|}$$

is convergent. Consequently in the whole $\zeta$-plane with the exception of the real integers ($\zeta' \neq$ integer), or in the whole $x$-plane with the exception of $x = a$, and $x' \neq a$,

$$\sum_{\lambda=-\infty}^{+\infty} (x_\lambda - x_\lambda')$$

is a uniformly convergent series of $x$.

## 4. *Example.*

If we take in (7) $\zeta' = -\zeta$, we get the function in the $\zeta$-plane

$$\eta = 2A\left(\frac{1}{\zeta} + \sum_{\lambda=1}^{+\infty} \frac{2\zeta}{\zeta^2 - \lambda^2}\right), \tag{8}$$

or, as is well known,

$$\eta = 2A\pi \cot \pi\zeta. \tag{9}$$

Transforming back to the $x$-plane, i. e., putting

$$\zeta = \frac{A}{x-a}, \quad \eta = \frac{A}{y-a},$$

we find

$$y = a + \frac{1}{2\pi} \tan \frac{\pi A}{x-a}. \tag{10}$$

This function is automorphic. If this is true, it must assume the same value when we replace $x$ by $x_\lambda$, or $x - a$ by $x_\lambda - a$. Now

$$\frac{A}{x_\lambda - a} = \frac{A}{x-a} + \lambda$$

and

$$\frac{\pi A}{x_\lambda - a} = \frac{\pi A}{x-a} + \lambda\pi.$$

From this we see immediately that (10) does not change its value under any substitution of the group.

In the $z$-plane defined by

$$\frac{A}{x-a} + c = z \tag{11}$$

we choose as the fundamental domain the strip between the axis of imaginaries and the line parallel to it at a distance equal to unity. Taking $c$ within this domain, let $z$ describe the line $z = c + it$, where $t$ is a real variable between $+\infty$ and $-\infty$. To the lines enclosing the strip correspond in the $x$-plane the two circles $K_{-1}$ and $K_1$ enclosing the fundamental domain $G$ in the $x$-plane. As $z$ describes the line $z = c + it$, $x$ describes the common tangent to $K_{-1}$ and $K_1$ at $a$, and according as $t$ approaches $+\infty$ or $-\infty$, $x$ approaches, within $G$, the point $a$ from opposite sides. To find the values of $y$ in (10) as $t = \pm\infty$, we substitute in (10) from (11)

$$\frac{A}{x-a} = z - c = it,$$

so that for points of the common tangent of $K_{-1}$ and $K_1$

$$y = a + \frac{1}{2\pi} \tan \pi i t. \tag{12}$$

But

$$\tan \phi = \frac{1}{i} \frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}},$$

so that

$$\lim_{t=\pm\infty} \{\tan \pi i t\} = \lim_{t=\pm\infty} \left\{ \frac{1}{i} \frac{e^{-\pi t} - e^{+\pi t}}{e^{-\pi t} + e^{+\pi t}} \right\} = \mp \frac{1}{i} = \pm i.$$

Thus, as $x$ approaches $a$ within $G$ from different sides, $y$ assumes at $a$ the values $a + \dfrac{i}{2\pi}$ and $a - \dfrac{i}{2\pi}$.

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# SOME PROPERTIES OF THE GROUP OF ISOMORPHISMS OF AN ABELIAN GROUP.

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(Read before the San Francisco Section of the American Mathematical Society, October 25, 1913.)

As the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, we shall assume that the order of the abelian group $G$ under consideration is $p^m$, $p$ being any prime number. Moreover, we shall confine our attention to a study of prop-