to unity. Taking c within this domain, let z describe the line $z = c + it$, where t is a real variable between $+\infty$ and $-\infty$. To the lines enclosing the strip correspond in the x-plane the two circles K_{-1} and K_1 enclosing the fundamental domain G in the x-plane. As z describes the line $z = c + it$, x describes the common tangent to K_{-1} and K_1 at a, and according as t approaches $+\infty$ or $-\infty$, x approaches, within G, the point a from opposite sides. To find the values of y in (10) as $t = \pm\infty$, we substitute in (10) from (11)

$$\frac{A}{x-a} = z - c = it,$$

so that for points of the common tangent of K_{-1} and K_1

$$y = a + \frac{1}{2\pi} \tan \pi i t. \tag{12}$$

But

$$\tan \phi = \frac{1}{i} \frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}},$$

so that

$$\lim_{t \doteq \pm\infty} \{\tan \pi i t\} = \lim_{t \doteq \pm\infty} \left\{ \frac{1}{i} \frac{e^{-\pi t} - e^{+\pi t}}{e^{-\pi t} + e^{+\pi t}} \right\} = \mp \frac{1}{i} = \pm i.$$

Thus, as x approaches a within G from different sides, y assumes at a the values $a + \dfrac{i}{2\pi}$ and $a - \dfrac{i}{2\pi}$.

UNIVERSITY OF ILLINOIS.

SOME PROPERTIES OF THE GROUP OF ISOMORPHISMS OF AN ABELIAN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, October 25, 1913.)

As the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, we shall assume that the order of the abelian group G under consideration is p^m, p being any prime number. Moreover, we shall confine our attention to a study of properties of the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms I of G. The number of these Sylow subgroups can easily be determined by means of the characteristic subgroups of G which involve only operators of order p besides identity. Let I' represent one of these Sylow subgroups and assume that the λ invariants of G are composed of the following numbers: λ_1 which are equal to p^{m_1}, λ_2 which are equal to p^{m_2}, $\cdots$, λ_r which are equal to p^{m_r}. Hence $\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_r m_r = m$; we shall assume that $m_1 > m_2 > \cdots > m_r$.

It is well known that the characteristic subgroups which are generated by operators of order p contained in G are as follows: One C_1 of order p^{λ_1} which is generated by all the operators of order p which are powers of the operators of highest order contained in G, and is known as the fundamental characteristic subgroup of G*; one C_2 of order $p^{\lambda_1+\lambda_2}$, which is composed of identity and the operators of order p which are powers of the operators of order p^{m_2} contained in G, and includes the preceding; etc. In general, there is one and only one such characteristic subgroup C_α of order $p^{\lambda_1+\lambda_2+\cdots+\lambda_\alpha}$, where α has any value from 1 to r, and this subgroup is generated by the operators of order p which are powers of those of order p^{m_α} contained in G, and it includes all those for smaller values of α.

The number of the Sylow subgroups of order $p^{m'}$ contained in I is clearly equal to the number of ways in which we can choose successively from $C_\alpha (\alpha = 1, 2, \cdots, r)$ one subgroup of each of the orders

$$p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}+1}, \quad p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}+2}, \quad \cdots, \quad p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha}-1}$$

such that each one of these subgroups includes all those which precede it and the first includes $C_{\alpha-1}$, C_0 being identity and λ_0 being 0. Hence the number of these Sylow subgroups contained in I is expressed by the following continued product:

$$\frac{p^{\lambda_1}-1}{p-1}\cdot\frac{p^{\lambda_1}-p}{p^2-p}\cdots\frac{p^{\lambda_1}-p^{\lambda_1-1}}{p^{\lambda_1}-p^{\lambda_1-1}}\cdot\frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1}}{p^{\lambda_1+1}-p^{\lambda_1}}\cdot\frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1+1}}{p^{\lambda_1+2}-p^{\lambda_1+1}}\cdots$$
$$\frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1+\lambda_2-1}}{p^{\lambda_1+\lambda_2}-p^{\lambda_1+\lambda_2-1}}\cdots\frac{p^{\lambda}-p^{\lambda-\lambda_r}}{p^{\lambda-\lambda_r+1}-p^{\lambda-\lambda_r}}$$
$$\frac{p^{\lambda}-p^{\lambda-\lambda_r+1}}{p^{\lambda-\lambda_r+2}-p^{\lambda-\lambda_r+1}}\cdots\frac{p^{\lambda}-p^{\lambda-1}}{p^{\lambda}-p^{\lambda-1}}.$$

* G. A. Miller, *Amer. Jour. of Mathematics*, vol. 27 (1905), p. 16.

This formula may evidently be reduced to the following much simpler form:

$$(p^{\lambda_1}-1)(p^{\lambda_1-1}-1)\cdots(p-1)(p^{\lambda_2}-1)(p^{\lambda_2-1}-1)$$
$$\cdots(p-1)\cdots(p^{\lambda_r}-1)(p^{\lambda_{r-1}}-1)\cdots(p-1)\div(p-1)^{\lambda}.$$

From the known theorem that the order of the group of isomorphisms of any abelian group is equal to the number of ways in which a set of independent generators may be selected, it results directly that the order of I is equal to the following continued product:

$$(p^m-p^{m-\lambda_1})(p^m-p^{m-\lambda_1+1})\cdots(p^m-p^{m-1})(p^{m-\lambda_1(m_1-m_2)}$$
$$-p^{m-\lambda_1(m_1-m_2)-\lambda_2})$$
$$p^{m-\lambda_1(m_1-m_2)}-p^{m-\lambda_1(m_1-m_2)-\lambda_2+1})\cdots(p^{m-\lambda_1(m_1-m_2)}-p^{m-\lambda_1(m_1-m_2)-1})$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$(p^{\lambda m_r}-p^{\lambda m_r-\lambda_r})(p^{\lambda m_r}-p^{\lambda m_r-\lambda_r+1})\cdots(p^{\lambda m_r}-p^{\lambda m_r-1}).$$

Hence a simple formula for the order of I is as follows:

$$(p^{\lambda_1}-1)(p^{\lambda_1-1}-1)\cdots(p-1)(p^{\lambda_2}-1)(p^{\lambda_2-1}-1)\cdots$$
$$(p-1)(p^{\lambda_r}-1)(p^{\lambda_r-1}-1)\cdots(p-1)p^{m'},$$

m' being equal to the following double sum:

$$\sum_{\alpha=1}^{\alpha=r}\sum_{\beta=1}^{\beta=\lambda_\alpha} m-\lambda_1(m_1-m_\alpha)-\lambda_2(m_2-m_\alpha)-\cdots-\lambda_{\alpha-1}(m_{\alpha-1}-m_\alpha)-\beta.$$

From the value of the order of I and the formula giving the number of the Sylow subgroups of order $p^{m'}$ contained in I, it results directly that each of these Sylow subgroups is transformed into itself by exactly $p^{m'}(p-1)^{\lambda}$ operators of I. Hence we have established the following theorem:

If an abelian group of order p^m has exactly λ invariants, the Sylow subgroup of order $p^{m'}$ in the group of isomorphisms of this abelian group is transformed into itself under this group of isomorphisms by a subgroup of order $p^{m'}(p-1)^{\lambda}$.

A direct corollary from this theorem is as follows:

A necessary and sufficient condition that the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms of an abelian group of order p^m are transformed into themselves only by their own operators under this group of isomorphisms is that $p=2$.

As a very special case of the formula giving the number of Sylow subgroups of order $p^{m'}$ in I, there results the following known theorem: A necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m contains only one Sylow subgroup of order $p^{m'}$ is that it contains no two equal invariants. It also results directly from this formula that a necessary and sufficient condition that an abelian group of order p^m has a group of isomorphisms whose order is of the form $p^{m'}$ is that $p = 2$ and that no two invariants of this abelian group are equal to each other.

Each of the given groups of order $p^{m'}(p-1)^\lambda$ which involve only one subgroup of order $p^{m'}$ includes $p-1$ of the $p^{m_1-1}(p-1)$ invariant operators of I. It also involves a subgroup of order $(p-1)^\lambda$, which is generated by λ operators of order $p-1$ such that each of these operators is commutative with all the independent generators of G except one, in a given set of independent generators, and that no two of these λ operators are non-commutative with the same independent generator. Hence this subgroup of order $(p-1)^\lambda$ is abelian and it is the direct product of λ cyclic groups of order $p-1$. It transforms into itself the cyclic subgroups generated by one, and only one, set of independent generators of G, and hence to each pair of Sylow subgroups of order $p^{m'}$ contained in I there correspond two sets of abelian subgroups of order $(p-1)^\lambda$ such that none of these subgroups is common to the two sets. In particular, we have established the following theorem:

If an abelian group of order p^m has λ invariants, each of the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms of this abelian group is transformed into itself by a group of order $(p-1)^\lambda$, which is the direct product of λ cyclic groups of order $p-1$.

Denoting by I' a Sylow subgroup of order $p^{m'}$ contained in I, we proceed to determine some restrictions on the orders of the operators of I'. It is well known that all these orders are divisors of p^{m-1}. Whenever G involves more than one invariant which is equal to p^{m_1} it is easy to prove that I' must involve at least one operator of order p^{m_1}. In fact, if S_1 and S_2 represent two independent generators of G and if each of these generators is of order p^{m_1}, I' includes an operator t which is commutative with each one of a set of independent generators of G, except S_2, and transforms S_2 into S_1S_2. As t^n transforms S_2 into $S_1^nS_2$, the order of t is clearly equal to p^{m_1}.

We proceed to prove that I' does not involve any operator whose order exceeds p^{m_1} whenever G does not involve more than p invariants which are equal to the same number. In fact, this result follows almost immediately from a method which has been employed to find the orders of operators in the group of isomorphisms of an abelian group.* According to this method the pth power of an operator of I' transforms every independent generator of G into itself multiplied by an operator whose order is less than the order of this independent generator. Hence there results the theorem:

If an abelian group of order p^m does not involve more than p invariants which are equal to each other, the group of isomorphisms of this abelian group involves no operator whose order is a power of p and exceeds the largest invariant of this abelian group.

UNIVERSITY OF ILLINOIS.

THEORY OF PRIME NUMBERS.

Handbuch der Lehre von der Verteilung der Primzahlen. Von Dr. EDMUND LANDAU, ordentlichem Professor der Mathematik an der königlichen Georg-August-Universität zu Göttingen. Vol. I, pp. xviii + 1–564; Vol. II, pp. ix + 565–961. Leipzig und Berlin, B. G. Teubner, 1909.

THE central problem in the prime number theory consists in proving that the number $\pi(x)$ of primes less than or equal to x may be represented asymptotically by the expression $x/\log x$, or

$$\lim_{x=\infty} \frac{\pi(x) \log x}{x} = 1. \tag{1}$$

This formula was conjectured by Legendre and Gauss at the end of the eighteenth century, but the first definite step toward a proof was taken by Tchebychef, who showed in 1851–52 that for sufficiently large values of x the quotient $\pi(x) \log x/x$ lies between two positive boundaries, one of them less and the other greater than unity. The next great advance is marked by Riemann's paper of 1859 "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse." His point of departure is the equation established by Euler

* G. A. Miller, BULLETIN, vol. 7 (1901), p. 351.