NOTE ON THE FREDHOLM DETERMINANT.

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THE theory of linear integral equations presents many analogies with the theory of linear algebraic equations; in fact the former may be regarded in a quite definite and accurate sense as a limiting case of the latter. As in various other mathematical theories concerned with limiting cases, two methods suggest themselves for the proof of theorems: one may go through the process of taking limits in the results of the algebraic theory—this method is typified by the early work of Hilbert* on integral equations; or one may use the algebraic theory merely to suggest theorems, which one then proves independently—this is the procedure in the fundamental paper of Fredholm.† While any sweeping statement comparing the two plans would be unwise, it seems reasonably clear that the second method will ordinarily be the more elegant.

In the course of a detailed study of certain properties of the kernel of an integral equation, Platrier‡ has recently given an expression for all higher minors of the determinant for any kernel in terms of the first minor and the determinant itself; his proof consists in carrying over by an accurate limit process the corresponding theorem for algebraic determinants. The same theorem had suggested itself to the present writer in studying mixed linear integral equations; as the proof devised by him, following directly from well known facts about the minors and involving no new passage to the limit, seems rather simpler, its presentation is worth while, in view of the importance of the theorem.§

It is assumed that the kernel $K(s, t)$ is a continuous function, $a \leqq s \leqq b$, $a \leqq t \leqq b$; integration always takes place between the limits a, b, which will not be written. The

* *Göttinger Nachrichten*, 1904, p. 4.

† *Acta Mathematica*, vol. 27 (1903), p. 365.

‡ *Liouville's Journal*, vol. 9 (1913), p. 249, equations (30), (31).

§ The writer learns that direct proofs of the theorem have also been found by Dr. T. H. Hildebrandt.

minors satisfy the identity (corresponding to the expansion of minors of an algebraic determinant according to columns)*

$$
\begin{aligned}
D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)& \\
= K(s_1, t_1)D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m)& \\
- K(s_1, t_2)D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m)& \\
+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot& \\
+ \lambda \int K(s_1, r)D(\lambda; r, s_2, \cdots, s_m; t_1, t_2, \cdots, t_m)dr.&
\end{aligned}
$$

Regarding all the quantities s, t, λ as constant except s_1, we have here an integral equation of the form

$$u(s_1) = f(s_1) + \lambda \int K(s_1, r)u(r)dr;$$

whether or not $D(\lambda) = 0$, we deduce that

$$D(\lambda)u(s_1) = D(\lambda)f(s_1) + \lambda \int D(\lambda; s_1, r)f(r)dr.$$

Thus

$$
\begin{aligned}
D(\lambda)D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)& \\
= [D(\lambda)K(s_1, t_1) + \lambda \int D(\lambda; s_1, r)K(r, t)dr]& \\
\times D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m)& \\
- [D(\lambda)K(s_1, t_2) + \lambda \int D(\lambda; s_1, r)K(r, t_2)dr]& \\
\times D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m)& \\
+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot&
\end{aligned}
$$

Simplifying each expression in brackets by the identity

$$D(\lambda)K(s, t) + \lambda \int D(\lambda; s, r)K(r, t)dr = D(\lambda; s, t),$$

we have

$$
\begin{aligned}
D(\lambda)D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)& \\
= D(\lambda; s_1, t_1)D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m)& \\
- D(\lambda; s_1, t_2)D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m)& \\
+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot&
\end{aligned}
\tag{1}
$$

The theorem to be proved now follows at once. In (1)

* Horn, Einführung in die Theorie der linearen partiellen Differentialgleichungen, p. 202.

take $m = 2$; we find

$$D(\lambda)D(\lambda; s_1, s_2; t_1, t_2)$$
$$= D(\lambda; s_1, t_1)D(\lambda; s_2, t_2) - D(\lambda; s_1, t_2)D(\lambda; s_2, t_1)$$
$$= \begin{vmatrix} D(\lambda; s_1, t_1) & D(\lambda; s_2, t_1) \\ D(\lambda; s_1, t_2) & D(\lambda; s_2, t_2) \end{vmatrix}.$$

This gives an expression for the minor of second order. Write (1) for $m = 3$, multiply by $D(\lambda)$, and replace the minor of second order by the value just obtained; we have a form for the minor of third order. Repeating this process, we find by mathematical induction the general formula

$$[D(\lambda)]^{m-1}D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)$$
$$(2) \qquad = \begin{vmatrix} D(\lambda; s_1, t_1) & D(\lambda; s_2, t_1) & \cdots & D(\lambda; s_m, t_1) \\ D(\lambda; s_1, t_2) & D(\lambda; s_2, t_2) & \cdots & D(\lambda; s_m, t_2) \\ \cdot & \cdot & \cdot & \cdot \\ D(\lambda; s_1, t_m) & D(\lambda; s_2, t_m) & \cdots & D(\lambda; s_m, t_m) \end{vmatrix},$$

which is the relation established by Platrier. In case $D(\lambda) \neq 0$ an equivalent form is of course obtained, as he points out, on dividing both sides by $[D(\lambda)]^{m-1}$,

$$D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)$$
$$(3) \qquad = D(\lambda) \begin{vmatrix} k(\lambda; s_1, t_1) & k(\lambda; s_2, t_1) & \cdots & k(\lambda; s_m, t_1) \\ k(\lambda; s_1, t_2) & k(\lambda; s_2, t_2) & \cdots & k(\lambda; s_m, t_2) \\ \cdot & \cdot & \cdot & \cdot \\ k(\lambda; s_1, t_m) & k(\lambda; s_2, t_m) & \cdots & k(\lambda; s_m, t_m) \end{vmatrix},$$

where $\lambda k(\lambda; s, t)$ is the resolvent to the kernel $\lambda K(s, t)$.

It is perhaps not superfluous to remark that the use of (2) as the *definition* of higher minors would lead to little or no simplification of the Fredholm theory. It would be necessary to show that the higher minors as thus defined are integral functions of the complex variable λ; the proof (thus required) that the m-rowed determinant, whose elements are values of the first minor, must possess at any principal value a zero of order $\geqq m - 1$, while not very difficult, is sufficiently so to preclude any substantial advantage in this mode of approach.

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