ON OVALS.

BY PROFESSOR TSURUICHI HAYASHI.

IN the *American Journal of Mathematics*, volume 35, number 4 (October, 1913), page 407,* Professor Arnold Emch has proved an interesting theorem that in every closed convex curve which is analytic throughout, at least one square may be inscribed; and in the BULLETIN, volume 20, number 1 (October, 1913), page 27,† he has proved that a closed convex analytic curve may be represented parametrically in the form

$$x = p + \frac{a}{2}\sqrt{2}\cos\left(\frac{2\pi t}{\omega} + \theta\right) + \sin\left(\frac{2\pi t}{\omega} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{\omega} - \frac{3\pi}{4}\right)f(t),$$

$$y = q + \frac{a}{2}\sqrt{2}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \sin\left(\frac{2\pi t}{\omega} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{\omega} - \frac{3\pi}{4}\right)g(t),$$

where $f(t)$, $g(t)$ are two uniform continuous functions for all values of t and with the same period ω, and p, q, a, and θ are certain constants. Pursuing his lines of investigation, I will prove here that about every closed convex curve which is analytic throughout (let me call this an oval simply) at least one square may be *circumscribed*, and will deduce a parametric representation of the curve in tangential coordinates.

Evidently a rectangle can be circumscribed about a given oval, one of whose four sides may take any direction whatever. If this rectangle, drawn quite at random, be a square, the question is solved. If it be not a square, then let its four sides be AB, BC, CD, DA, and let the points of contact of these sides be P, Q, R, S in order respectively. Now the ratio $BC : AB$ (k say) is not equal to unity. If it be greater than unity, the ratio $CD : BC$ ($1/k$) is less than unity. Hence when the point of contact passes from P to Q along the curve

* "Some properties of closed convex curves in a plane."
† "On closed continuous curves."

and, what is the same thing, when the direction of the side AB becomes that of the side BC, the ratio of the two neighboring sides changes from k (> 1) to $1/k$ (< 1). Therefore by the principle of continuity, this ratio must become equal to unity at least once during this change. At that time the rectangle becomes a square.

Let us next seek a parametrical representation of the curve in tangential coordinates from this consideration, by connecting the tangential coordinates (λ, μ) and the point coordinates (x, y) by the equation

$$\lambda x + \mu y = 1.$$

Draw a square with diagonal $2a$, symmetrically situated with respect to the x and y axes, and having its vertices on the axes, so that the four sides are

$$x + y = a, \qquad -x + y = a, \qquad -x - y = a, \qquad x - y = a,$$

the tangential coordinates being

$$\left(\frac{1}{a}, \frac{1}{a}\right), \quad \left(-\frac{1}{a}, \frac{1}{a}\right), \quad \left(-\frac{1}{a}, -\frac{1}{a}\right), \quad \left(\frac{1}{a}, -\frac{1}{a}\right),$$

respectively. Then the parametrical representation of any oval through the vertices of this square in the tangential coordinates must be of the form

$$\lambda = \frac{\sqrt{2}}{a} \cos \frac{2\pi t}{\omega} + \cos \frac{4\pi t}{\omega} \varphi(t),^*$$

$$\mu = \frac{\sqrt{2}}{a} \sin \frac{2\pi t}{\omega} + \cos \frac{4\pi t}{\omega} \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are two uniform continuous functions for all values of t, and with the same period ω, the parameters corresponding to the sides of the square being $t_k = (2k + 1)\dfrac{\omega}{8}$ ($k = 0, 1, 2, 3$). The process of obtaining this representation is quite the same as that used by Professor Emch in his paper above cited.

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* Professor Emch has used for his purpose the more complicated factor

$$\sin\left(\frac{2\pi t}{a} - \frac{\pi}{4}\right) \sin\left(\frac{2\pi t}{\omega} - \frac{3\pi}{4}\right)$$

than the factor $\cos 4\pi t/\omega$ here used, though the former is simpler than the product of four sines first used by him (see BULLETIN, vol. 19, No. 5 (February, 1913), pp. 221–222).

Applying to the points (x, y) of the cartesian plane a combined rotation and translation θ, p, q, so that the coordinates of the points before and after the motion are connected by

$$X = p + x\cos\theta - y\sin\theta,$$
$$Y = q + x\sin\theta + y\cos\theta$$

or

$$x = (X - p)\cos\theta + (Y - q)\sin\theta,$$
$$y = -(X - p)\sin\theta + (Y - q)\cos\theta,$$

then the straight line

$$\lambda x + \mu y = 1$$

becomes

$$\lambda\{(X - p)\cos\theta + (Y - q)\sin\theta\} + \mu\{-(X - p)\sin\theta + (Y - q)\cos\theta\} = 1,$$

i. e.,

$$\Lambda X + \mathrm{M} Y = 1,$$

if we put

$$\Lambda = \frac{\lambda\cos\theta - \mu\sin\theta}{1 + p(\lambda\cos\theta - \mu\sin\theta) + q(\lambda\sin\theta + \mu\cos\theta)},$$

$$\mathrm{M} = \frac{\lambda\sin\theta + \mu\cos\theta}{1 + p(\lambda\cos\theta - \mu\sin\theta) + q(\lambda\sin\theta + \mu\cos\theta)}.$$

Hence, replacing λ and μ by their values, we get the tangential coordinates Λ and M of the oval in terms of the parameter t, in the form

$$\Lambda = \left\{\frac{\sqrt{2}}{a}\cos\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega} f(t)\right\} \div \mathrm{N},$$

$$\mathrm{M} = \left\{\frac{\sqrt{2}}{a}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega} g(t)\right\} \div \mathrm{N},$$

$$\mathrm{N} = 1 + p\left\{\frac{\sqrt{2}}{a}\cos\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega} f(t)\right\} + q\left\{\frac{\sqrt{2}}{a}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega} g(t)\right\},$$

where

$$f(t) = \varphi(t)\cos\theta - \psi(t)\sin\theta, \quad g(t) = \varphi(t)\sin\theta + \psi(t)\cos\theta.$$

Therefore this form must be the required parametrical representation of *any* oval in tangential coordinates, if we choose the unit of length properly.*

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ON THE CLASS OF DOUBLY TRANSITIVE GROUPS.

BY PROFESSOR W. A. MANNING.

(Read before the San Francisco Section of the American Mathematical Society, October 25, 1913.)

THE class $u (u > 3)$ of a doubly transitive group of degree n is, according to Bochert,† greater than $\frac{1}{3}n - \frac{2}{3}\sqrt{n}$. If we confine our attention however to those doubly transitive groups in which one of the substitutions of lowest degree is of order 2, it appears that the class is greater than $\frac{1}{2}n - \frac{1}{2}\sqrt{n} - 1$. The proof of this statement rests essentially upon the following

LEMMA. *The degree of a diedral group of class u generated by two non-commutative substitutions of order* 2 *and degree u is at most* $\frac{3}{2}u$.

Let s and t be the two substitutions in question, and let the order of their product be $N = 2^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where p_1, p_2, $\cdots$ are distinct odd primes. The transitive constituents of $\{s, t\}$ may be arranged as follows:

s has m_1 cycles displacing letters not in t, and t has m_2 cycles displacing letters not in s; there are x_i regular constituents of order X_i, with a generator in both s and t (thus common cycles of s and t are explicitly included, while the preceding type of constituent of degree and order 2 is excluded); there are y_j non-regular constituents of degree Y_j and order $2Y_j$, Y_j an odd number; there are y_k' non-regular constituents of degree Y_k' and order $2Y_k'$, Y_k' even, with the generator of degree Y_k' in s, and the generator of degree $Y_k' - 2$ in t; in like manner there are y_k'' constituents of the order Y_k' with $Y_k' - 2$ letters in s and Y_k' letters in t. Since transitive

* Subsequently I have proved that an infinite number of cubes may be circumscribed about an ovoid body. The proof and application of this theorem will be published in the *Science Reports* of the Tôhoku University, Sendai, vol. 3, no. 4.

† Bochert, *Math. Annalen*, vol. 49 (1897), p. 131.