THE APRIL MEETING OF THE SOCIETY AT CHICAGO.

The thirty-fifth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 2–3, 1915, it being the fourth regular meeting of the Society at Chicago. Seventy-five persons were in attendance upon the various sessions, including the following fifty-three members of the Society:

Professor G. A. Bliss, Professor P. P. Boyd, Professor J. W. Bradshaw, Dr. Josephine E. Burns, Professor R. D. Carmichael, Professor A. F. Carpenter, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Mr. L. C. Cox, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Mr. C. R. Dines, Professor Arnold Dresden, Professor W. B. Ford, Professor A. B. Frizell, Professor E. R. Hedrick, Professor O. D. Kellogg, Dr. A. J. Kempner, Professor Kurt Laves, Professor C. E. Love, Dr. W. V. Lovitt, Professor A. C. Lunn, Dr. E. B. Lytle, Professor Malcolm McNeill, Professor W. D. MacMillan, Dr. T. E. Mason, Professor G. A. Miller, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor S. W. Reaves, Professor H. L. Rietz, Mr. George Rutledge, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Professor Mary E. Wells, Professor W. D. A. Westfall, Professor E. J. Wilczynski, Professor D. T. Wilson, Professor A. E. Young, Professor J. W. A. Young, Mr. C. H. Yeaton, Professor Alexander Ziwet.

Professor F. R. Moulton, Vice-President of the Society, presided at the sessions on Friday morning and Saturday afternoon, and Professor E. J. Wilczynski at those on Friday afternoon and Saturday morning.

The social gathering and dinner on Friday evening was attended by forty-seven members.

The following papers were presented at this meeting:

(1) Professor C. H. Sisam: "On sextic surfaces having a nodal curve of order nine."

(2) Professor A. F. CARPENTER: "A theorem on the flecnode curves of a ruled surface."

(3) Professor C. T. SULLIVAN: "Analytic characterization of surfaces having a degenerate directrix quadric."

(4) Professor L. E. DICKSON: "Modular quartic curves."

(5) Dr. W. V. LOVITT: "A type of singular points for a transformation on three variables."

(6) Dr. W. V. LOVITT: "Singularities of contact transformations in two-space."

(7) Professor E. J. WILCZYNSKI: "Relations between the theory of congruences and the theory of surfaces."

(8) Address by former chairman of the Chicago Section Professor E. H. MOORE: "Report on integral equations in general analysis."

(9) Dr. G. E. WAHLIN: "A new development of the theory of algebraic domains."

(10) Professor G. A. BLISS: "Jacobi's condition for problems of the calculus of variations."

(11) Professor C. E. LOVE: "On linear difference and differential equations."

(12) Mr. W. L. HART: "Theorems on functions of infinitely many variables" (preliminary report).

(13) Professor F. R. MOULTON: "The solution of an infinite system of differential equations" (preliminary report).

(14) Professor C. N. MOORE: "On the summability of certain types of series."

(15) Professor R. D. CARMICHAEL: "On the solutions of linear homogeneous difference equations."

(16) Professor R. D. CARMICHAEL: "On certain problems in diophantine analysis."

(17) Professor E. B. STOUFFER: "On seminvariants of linear homogeneous differential equations."

(18) Professor W. B. FORD: "On the representation of arbitrary functions by definite integrals."

(19) Professor W. D. MACMILLAN: "The convergence of the power series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{a_{ij}}{i - j\gamma} x_1^i x_2^j$."

(20) Dr. E. W. CHITTENDEN: "On equivalence of K relations."

(21) Dr. E. W. CHITTENDEN: "On the existence of continuous functions."

(22) Professor L. E. DICKSON: "Historical note on the

proof of the quadratic reciprocity law in a posthumous paper by Gauss."

(23) Professor G. A. MILLER: "Independent generators of a group of finite order."

(24) Professor A. B. FRIZELL: "Well ordered sets of infinite permutations."

(25) Mr. C. R. DINES: "On the Hilbert-Schmidt theory in general analysis and the generalization of a theorem by Picard."

(26) Mr. C. R. DINES: "On the instance suggested by the analogy of the sphere and the ellipsoid as regards positive kernels."

(27) Professor E. J. MOULTON: "On the deviation of a rotating compressible fluid mass from a true spheroid."

(28) Professor R. D. CARMICHAEL: "On the use of functional equations in diophantine analysis."

The papers of Professors Sullivan, Stouffer, and Miller, and the first paper by Dr. Lovitt were read by title. Mr. Hart was introduced by Professors Moore and Moulton. Abstracts of the papers follow, the numbers corresponding to those in the list of titles above.

1. In this paper, Professor Sisam points out some fundamental properties of the unruled sextic surface having a nodal curve of order nine, and of the sextic surface in space of six dimensions of which it is the projection.

2. By determining the conditions satisfied by the coordinates of a plane fixed in space, when referred to a moving tetrahedron of reference, Professor Carpenter proves the following theorem: Let an arbitrary fixed plane P cut the generators of a ruled surface S and let π be the pole of P with respect to the quadric osculating S along a generator g. Then the tangent at π to the locus of π cuts g in a point which is harmonically separated from the point in which P cuts g by the flecnodes of g. A special case of this theorem is stated by M. A. Demoulin in the *Comptes Rendus*, June, 1908, page 1381.

3. The integral surfaces of a completely integrable system of partial differential equations

$$\frac{\partial^2 y}{\partial u^2} + 2b\frac{\partial y}{\partial v} + fy = 0, \quad \frac{\partial^2 y}{\partial v^2} + 2a'\frac{\partial y}{\partial u} + gy = 0,$$

where the fundamental invariants a' and b are characterized by the value

$$a' = b = \frac{1}{2}\frac{\sqrt{\left(\frac{dU}{dy}\right)\left(\frac{dV}{dv}\right)}}{(U+V)},$$

constitute the class of surfaces whose asymptotic curves belong to linear complexes. In a former paper (in volume 15 of the *Transactions*), Dr. Sullivan studied these surfaces and showed that they were the envelopes of either of two one-parameter families of ruled surfaces $[R(u), R(v)]$ with straight line directrices; and that the loci of the directrices of $[R(u), R(v)]$ were complementary reguli of a quadric surface (directrix quadric).

The present paper is devoted to the exceptional case where the directrix quadric degenerates into two planes, distinct or coincident. It is shown that the nature of the directrix quadric is completely characterized by the forms of the two invariants θ/a'^2 and θ'/b^2, which played such a fundamental rôle in the determination of the canonical forms of the seminvariants $f(u, v)$ and $g(u, v)$.

It is found that the most general forms of these invariants are:

1. Directrix quadric non-degenerate,

$$\left(\frac{\theta}{a'^2}\right) = \frac{a_1U^2 + b_1U + c}{(dU/du)}, \quad \left(\frac{\theta'}{b^2}\right) = \frac{a_1V^2 - b_1V + c}{(dV/dv)},$$

where a_1, b_1 and c are constants.

2. Directrix quadric degenerates into two distinct planes,

$$a_1 = b_1 = 0, \quad c \neq 0.$$

3. Directrix quadric degenerates into two coincident planes,

$$a_1 = b_1 = c = 0.$$

The plane of case 3 is identical with that determined by the plane net of lines of the degenerate directrix congruence of the first kind discussed in section 7 of the paper cited above. In this case the finite equations of the surfaces have been obtained by Professor Wilczynski ("Flächen mit unbestimmten Direktrixkurven," *Mathematische Annalen*, December, 1914).

4. Professor Dickson's paper will appear in the current volume of the *American Journal of Mathematics*.

5. It is the purpose of Dr. Lovitt's first paper to discuss from a geometrical standpoint, as far as possible, the character of a transformation

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

in the neighborhood of a special type of point at which the Jacobian of the transformation vanishes.

The initial assumptions are such that attention is confined to non-singular points both of the surface defined by the Jacobian of the transformation and of the transform of this surface by means of equations (1). In section two it is shown that under our assumptions there is a one-to-one correspondence between either of two regions in uvw-space on opposite sides of the Jacobian surface J and their transforms in the xyz-space, the transformed regions being on the same side of a surface J_1 which is the transform of the surface J by means of equations (1). Moreover each of the transformed regions is a connected region possessing interior points. The equations of transformation define inverse functions which are continuous at points of the xyz-space which are on the surface J_1. In section three most of the results of the preceding section are extended to a transformation of a neighborhood of a portion of a singular surface not necessarily restricted to a neighborhood of a particular point. In section four most of the results of the preceding section are extended to a transformation in n variables.

6. In his second paper, Dr. Lovitt studies the effect of the transformation

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p)$$

upon the contact of curves

$$(1) \qquad x = f(t), \quad y = g(t), \quad y' = g'/f' = h(t)$$

in the neighborhood of a set of values $(x, y, p = x_0, y_0, p_0)$ for which the functional determinant of the transformation vanishes, but the matrix of the determinant is of rank two. Let the transformed curves be given by

$$x_1 = f_1(t), \quad y_1 = g_1(t), \quad y_1' = g_1'/f_1' = h_1(t),$$

where f_1, g_1, h_1 are defined by means of the equations of transformation and equations (1). Let us designate as critical curves that family of curves which yield the values of $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ necessary to make

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \cdots, k - 1,$$

while $[f_1^{(j)}]^2 + [h_1^{(j)}]^2 \neq 0$.

Then the necessary and sufficient condition that two of the critical curves shall be transformed into curves with second order contact is that the critical curves have contact of order $k + 1$.

7. In a paper presented to the Society at Chicago in December, 1914, Professor Wilczynski introduced a number of new concepts into the theory of congruences, among which the axis and ray congruences, and the axis and ray curves were especially important. On the other hand he has introduced into the theory of surfaces the notion of the directrix congruences and the directrix curves. In the present paper the theory of congruences is enriched by formulas and theorems expressing the relations which exist between the directrix curves and directrix congruences of the focal surface, and the other properties of the congruence.

9. In volume 144 of *Crelle's Journal* Hensel published a paper entitled "Ueber die Grundlagen einer neuen Theorie der quadratischen Zahlkörper." Dr. Wahlin's paper is a generalization of this development to any algebraic domain of degree n.

In the first place the ring $R(q, \alpha)$, consisting of the g-adic algebraic numbers determined by the root α of an equation of degree n, is considered. It is shown how the study of these numbers can be reduced to the case where g is a rational prime p and we have the ring $R(p, \alpha)$.

A method is then devised by means of which the further study of this ring can be reduced to the study of certain domains constructed out of the numbers of $R(p, \alpha)$ with a different condition for equality.

10. There are two well-known methods of deducing Jacobi's necessary condition in the calculus of variations. One is a geometric proof not easily extensible to higher spaces, which excludes important exceptional cases; the other depends upon complicated manipulations of the second variation. For the problem in parametric form in the plane the reduction

of the second variation was devised by Weierstrass and is a remarkable piece of analysis. It is, however, very artificial, and also unnecessary if one has in view only the proof of Jacobi's condition. For problems in parametric form in higher spaces a discussion of the second variation has been made by von Escherich by methods in part quite unsymmetrical. The lack of symmetry is due to the division of an arc

$$y_i = y_i(t) \quad (t_1 \leqq t \leqq t_2;\ i = 1, 2, \cdots, n)$$

into a finite number of pieces on each of which at least one of the derivatives $y_i'(t)$ is different from zero, a device which leads to inelegant complications, though his results are symmetric in form. In the paper of Professor Bliss a proof of Jacobi's condition is given which applies with equal simplicity to the parametric case in the plane or in higher spaces. It makes use of the second variation without reduction of any sort, is symmetric in all the variables, and includes the exceptional cases which probably would with difficulty be covered by an extension to higher spaces of the geometric proof mentioned above.

11. The first part of Professor Love's paper is an extension of certain theorems of Ford* on linear difference equations. The behavior, for large positive integral values of x, of n linearly independent solutions of the equation

$$Y(x) = [a_0 + \alpha_0(x)]y(x+n) + [a_1 + \alpha_1(x)]y(x+n-1) \\ + \cdots + [a_n + \alpha_n(x)]y(x) = 0$$

is determined, provided the functions $\alpha_0(x), \alpha_1(x), \cdots, \alpha_n(x)$ vanish at infinity to a sufficiently high order. It should be noted that no restriction whatever is laid upon the roots of the characteristic equation

$$a_0\mu^n + a_1\mu^{n-1} + \cdots + a_n = 0,$$

except that a_0 and a_n are assumed different from zero.

The second part of the paper contains the parallel investigation for linear differential equations. The same problem under narrower conditions has been treated by Dini.†

* *Annali di Matematica*, ser. 3, vol. 13 (1907), pp. 263–328.
† *Annali di Matematica*, ser. 3, vol. 2 (1899), pp. 297–324; *ibid.*, vol. 3 (1900), pp. 125–183.

In each part of the paper the results obtained are also applied to the study of the non-homogeneous equation

$$Y(x) = X(x),$$

under suitable restrictions on the function $X(x)$.

12. By a function of infinitely many real variables will be understood a function $f(\xi)$ where ξ represents $(x_1, x_2, \cdots)$. In much of the current work, the function f is defined for values ξ for which $\Sigma_{i=1}^{i=\infty} | x_i^2 |$ converges. In Mr. Hart's paper the functions considered are supposed to be defined for values ξ satisfying

$$M_1^{(i)} \leqq x_i \leqq M_2^{(i)} \quad (M_1^{(i)}, M_2^{(i)} \leqq M; i = 1, 2, \cdots).$$

The function f is said to be completely continuous at the point ξ_0 if, whenever

$$\lim_{n=\infty} x_{i,\,n} = x_{i,\,0} \qquad (i = 1, 2, \cdots),$$

then

$$\lim_{n=\infty} f(\xi_n) = f(\xi_0) \quad (\xi_n = x_{1,\,n}, x_{2,\,n}, \cdots).$$

In addition to certain general theorems on such continuous functions, among which is included an extension of the mean value theorem, there is considered the solution, for ξ in terms of u, of the infinite system of equations

$$f_i(\eta) = 0 \quad (\eta = u, x_1, x_2, \cdots; i = 1, 2, \cdots), \tag{1}$$

where the values of η lie in the region R defined by

$$| \eta - \eta_0 | \leqq d_0.$$

The fundamental theorem of implicit function theory is obtained for (1) by a method of successive approximations.

13. Professor Moulton treats an infinite system of differential equations of the analytic type, and, under suitable hypotheses, establishes the existence of an analytic solution. The solution is also studied as a function of the parameters which may be involved and of the initial conditions. The paper is incomplete in that the details of the extension of the solution to the boundary of the region where the initial equations are defined and regular have not been fully worked out.

14. Chapman has obtained sufficient conditions that a series summable (Ck) for any $k > 0$ should be reduced to convergence when certain convergence factors are introduced into the terms of the series. Somewhat earlier G. H. Hardy discussed the conditions under which a series summable (Ck), where k is a positive integer, should remain summable (Ck) when the terms are multiplied by factors of a certain type. In Professor Moore's paper sufficient conditions are obtained that a series summable (Ck) for any $k > 0$ should remain summable (Ck) when certain factors are introduced into the terms.

15. In his first paper Professor Carmichael gives a new method of deriving the general existence theorems for the case of the linear homogeneous difference equation

$$f(x + n) + a_1(x)f(x + n - 1) + \cdots + a_n(x)f(x) = 0$$

in which the coefficients $a_1(x), \cdots, a_n(x)$ have the character of rational functions at infinity. By means of the descending formal power series solutions which this equation is known to possess in general a separation of it into two members is effected so that the method of successive approximation is available for deriving a single solution. By repeated use of a method of transformation the remaining solutions of two fundamental sets are then obtained. The analytic character of the functions in one of these sets and their asymptotic character at infinity are determined.

Of the previously developed methods for dealing with the problem of this paper that of the author's dissertation is most closely related to the present one, but the two methods are essentially distinct.

16. By means of a class of algebraic relations due to Fermat and Lagrange and developed in part by Legendre, Professor Carmichael, in his second paper, suggests a general method by which large classes of problems in diophantine analysis become amenable to systematic treatment. The method is applied to the resolution of a considerable number of diophantine equations of interest. The essential element of the method may be described thus: The problem of solving a diophantine equation consists essentially in determining numbers of a certain class (those defined by the form of one member) so that they shall have an additional property (that determined

by the form of the other member). The method suggested and employed in the present paper is suitably to extend the set of numbers defined by the form in one member (sometimes by those in both) so that the extended set shall have the property that the product of any two numbers in the set shall also be in the set. This gives rise to a generalized problem. The latter is solved and the result is then specialized so as to afford the solution of the original problem. This method, though very simple in its characteristics, leads to a considerable number of results of interest.

17. Consider the system of linear homogeneous differential equations

$$y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0, \quad (i = 1, 2, \cdots n),$$

where p_{ik} and q_{ik} are functions of the independent variable x. The most general transformations leaving such a system unchanged in form are given by

$$(1) \qquad y_i = \sum_{k=1}^{n} \alpha_{ik}\bar{y}_k, \qquad | \alpha_{ik} | \neq 0, \quad (i = 1, 2, \cdots n),$$

$$(2) \qquad x = \xi(x),$$

where α_{ik} and ξ are arbitrary functions of x. A function of p_{ik} and q_{ik} and their derivatives which has the same value for the given system as for any system obtained from it by the transformation (1) is called a seminvariant. Professor Stouffer obtains for each value of n a single seminvariant from which the complete system for that value of n may be obtained by successive applications of certain operators. These operators are similar to the Aronhold operator for algebraic invariants. This method of obtaining the seminvariants avoids the solution of complicated systems of partial differential equations.

18. Let $f(x)$ be an arbitrary function of the real variable x defined throughout the interval $(-\pi, \pi)$. Having assigned the special value α $(-\pi < \alpha < \pi)$ to x, let the Fourier series for $f(\alpha)$ be formed. The sum of its first $n + 1$ terms may be put in the form

$$(1) \qquad S(n, \alpha) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\psi(n, x - \alpha)dx,$$

where

$$\psi(n, x - \alpha) = \frac{\sin \frac{1}{2}(2n + 1)(x - \alpha)}{2 \sin \frac{1}{2}(x - \alpha)},$$

and it is a familiar fact that we then have, at least in general,

$$(2) \qquad \lim_{n=\infty} S(n, \alpha) = \frac{f(\alpha - 0) + f(\alpha + 0)}{2}.$$

The following general problem is thus suggested: Given an arbitrary function $f(x)$ defined throughout any finite interval (a, b) and consider the integral

$$(3) \qquad I(n, \alpha) = \int_a^b f(x)\varphi(n, x - \alpha)dx,$$

where $\varphi(n, x - \alpha)$ is now regarded as an unknown function to be so determined that we shall have, analogously to (2),

$$(4) \qquad \lim_{n=\infty} I(n, \alpha) = \frac{f(\alpha - 0) + f(\alpha + 0)}{2} \qquad (a < \alpha < b).$$

What conditions upon $\varphi(n, x - \alpha)$ will suffice to insure (4)? This problem was first considered and answered to some extent by Du Bois-Reymond, later it was treated in great detail by Dini in his "Serie di Fourier," and of late years it has been the object of a paper by Hobson.* The conditions in question have thus been in large measure determined, but it does not appear that explicit forms for $\varphi(x, n - \alpha)$ that satisfy such conditions have been worked out to any appreciable extent. In this connection, Professor Ford's paper shows how an infinity of such functions $\varphi(n, x - \alpha)$ may be explicitly obtained.

19. In this paper by Professor MacMillan it is shown that if the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x_1^i x_2^j$$

is convergent for $| x_1 | < x_{10}$, $| x_2 | < x_{20}$, then the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_{ij}}{i - j\gamma} x_1^i x_2^j$$

also is convergent if $| x_1 | < x_{10}$, $| x_2 | < x_{20}$, and γ is a positive irrational number which satisfies certain mild conditions.

* *Proc. London Math. Soc.*, vol. 6 (1908), pp. 349–374.

Two applications of these series are given. The first application is to the function

$$W = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda^i \mu^j}{i - j\gamma},$$

for which the positive real axis is a line of essential singularities. It is shown that this function is continuous across this axis at such irrational points. The second application is to the existence of a solution of the linear partial differential equation

$$x_1 \frac{\partial \varphi}{\partial x_1} - \gamma x_2 \frac{\partial \varphi}{\partial x_2} = p_1 \varphi + p_2,$$

where p_1 and p_2 are convergent ordinary power series in x_1 and x_2. It is shown that solutions exist as ordinary power series in x_1 and x_2.

20. For a class $\mathfrak{P}$ relations Kp_1p_2m, $\overline{K}p_1p_2m$ are supposed to be defined. These relations are of the type K_2 of Moore's Introduction to General Analysis. Dr. Chittenden defines equivalence for the relations K, $\overline{K}$ and shows that for any unsymmetric K possessing the properties 167 as defined by T. H. Hildebrandt* a symmetric equivalent relation $\overline{K}$ may be defined. The theory of classes $\mathfrak{P}$ and functions on classes $\mathfrak{P}$ as developed by Hildebrandt for an unsymmetric K^{167} relation is equivalent to the corresponding theory for symmetric K relations.

21. Hans Hahn has shown that there exists a non-constant continuous function on every class $\mathfrak{P}$ which admits a definition of voisinage.† Dr. Chittenden shows by modifying the methods of Hahn that this result may be extended to classes which admit a definition of distance subject to the condition $L_n(p_n, q_n) = 0$, $L_n(p_n, p) = 0$ imply $L_n(q_n, p) = 0$. It follows, as shown by Hahn, that, if limit is unique, a necessary and sufficient condition that every continuous function on $\mathfrak{P}$ be bounded and attain its bounds is that $\mathfrak{P}$ be extremal. This result is extended to the case of non-unique limits.

* *American Journal of Mathematics*, vol. 34 (1912), pp. 243–4.
† *Monatshefte für Mathematik und Physik*, vol. 19 (1908), p. 251–5.

22. The seventh proof by Gauss of the quadratic reciprocity theorem occurs in his posthumous paper, Werke 2, 1863, pages 233–5 (Gauss-Maser, pages 623–4). Gauss left only a meager outline of a possible proof, not revised by him for publication. In the version by Bachmann, Niedere Zahlentheorie, 1, 1902, pages 396–9, resort is had (page 399) a second time to the cyclotomic theory, whereas one needs only the fact that in all cases $S(i)$ and $S_1(i)$ are the roots of the congruence at the bottom of page 398; moreover, Bachmann introduces Galois imaginaries, whereas their use was carefully avoided by Gauss (Disquisitiones Arithmeticae, article 338; Werke, 2, page 217). The object of the note by Professor Dickson is to complete the proof in accord with the spirit of the MS. of Gauss. In *Crelle*, 19, 1839, pages 299–306, Schonemann gave a sketch (neither very clear nor full) of a similar proof based on the reality of the roots of the quadratic congruence corresponding to the cyclotomic equation for two periods. Another proof similar to that by Gauss and published three years prior to his is that by V. A. Lebesgue, *Comptes Rendus*, 51, 1860, pages 9–13; it is based on Kummer's theorem on the reality of the roots of the congruence corresponding to a cyclotomic equation for e periods.

23. The term independent generator has been used mainly with reference to abelian groups. Hence it has been commonly used with the restricted meaning that the group generated by any number of a set of these generators has only identity in common with the group generated by the rest of those in the same set. With this restricted meaning Frobenius and Stickelberger proved in 1879 that the number of the independent generators is an invariant of any abelian group whose order is the power of a prime number. It has been proved recently that the number of these independent generators is an invariant of *every* group whose order is a power of a prime when the independent generators are defined as follows: A set of operators $s_1, s_2, \cdots, s_\lambda$ belonging to the group G is called a set of independent generators of G provided that these λ operators generate G but no $\lambda - 1$ of them generate G.

Among the theorems established by Professor Miller in the present paper are the following: A necessary and sufficient condition that a group G contains at least one set of independent generators composed of as many operators as there are prime factors in the order of the group, is that each of the Sylow subgroups of G is abelian and is generated by a set of operators of prime order each of which is transformed only into powers of itself by every Sylow subgroup whose order is a power of a smaller prime number. The number of the groups of order p^m, p being any prime number, which possess a set of $m-1$ independent generators is either $\frac{3}{2}(m-1)$ or $\frac{3}{2}(m-2)+1$ according as m is odd or even.

24. By a process used in a paper read before the Society at New York, February, 1915, Professor Frizell proves that all well ordered types of order are comprehended in those furnished by the permutations of a simply infinite set. That is, given a well ordered set w, there exists, among the permutations of an ω-series, a set ordinally similar to w.

25. Professor Moore has shown that the general Hilbert-Schmidt theory of the integral equation for the complex-valued hermitian kernel may be secured on the foundation Σ_5; viz.,

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}^{LCDD_0R \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHPP_0 \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}}),$$

where $\mathfrak{A}$ is the class of all real or of all complex numbers, $\mathfrak{P}$ is a general class of general elements forming a general range, $\mathfrak{M}$ is a class of single-valued functions on the range $\mathfrak{P}$ to $\mathfrak{A}$ with the properties $LCDD_0R$, $\mathfrak{K}$ is defined as the *-composite of $\mathfrak{M}$ with itself, and J is a general functional operation having properties as indicated. The definite property P_0 of the functional operation is the property which, in the classical instance of integration states that if, for a continuous function ξ, we have $\int_0^1 \xi(s)\bar{\xi}(s)ds = 0$, then $\xi(s)$ is identically zero. Mr. Dines shows that the theory as to the existence of characteristic functions and numbers of a hermitian kernel κ may be obtained on a foundation in which the functional operation is not restricted by postulation of the property P_0. This permits us then to include as an instance that of integration in which the class $\mathfrak{M}$ is taken as the class of all real-valued, bounded functions, integrable with their squares in the sense of Lebesgue. The relation, κ not identically zero, in the classical instance is replaced by the relation $J_{23\ 41}{}^2\kappa\kappa \neq 0$, and the theory is in general the same. The expansion theorem of Hilbert is stated as giving the expansion of $J^2\alpha\kappa\beta$ where α and β are not necessarily in the class $\mathfrak{M}$ but are functions of a class $\mathfrak{N}$ which contains $\mathfrak{M}$ as a subclass, J being operative on $(\mathfrak{N}\mathfrak{N})_L$ to $\mathfrak{A}$. For example, the class $\mathfrak{N}$ may be taken as the class of all functions, whether bounded or not, integrable with their squares in the sense of Lebesgue.

The unsymmetric kernel is then discussed in a generalization in which not only is the functional operation generalized but also the kernel function is no longer on a composite range necessarily formed by one range with itself but on a composite range formed from two conceptually distinct ranges.

Using these results, Mr. Dines secures a generalization of a theorem due to Picard as to conditions for solvability of an integral equation of the first kind and gives an application to the geometry of a function space.

26. As an instance of the basis Σ_5, Professor Moore has cited that suggested by the analogy of the sphere and the ellipsoid. In this instance, given a function ω, a new functional operation $J\omega$ is obtained from that of Σ_5 by defining the functional operation, operating on a function κ, as $J_{13\ 42}{}^2\kappa\omega$. Mercer has shown the importance of the positive kernel as to expansion in a uniformly convergent series of characteristic functions. Mr. Dines discusses the positive kernel in the instance cited above. Conditions on a hermitian function ω are obtained sufficient to secure the equivalence for every hermitian function κ of the properties, positive as to J and positive as to J_ω. There follows a discussion of conditions equivalent to these. The latter part of the work is suggested by that of Fischer on the equivalence of the notions, general and closed, as applied to a unitary and orthogonal set of functions from the class of all functions integrable with their squares in the sense of Lebesgue. The foundation for the discussion is Σ_7, defined as follows:

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0RC_J \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{M}^{LCDD_0RB_0\mathfrak{N}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHP \text{ on } \binom{(\mathfrak{N}\mathfrak{N})L}{\mathfrak{K}} \text{ to } \mathfrak{A}}).$$

This differs from the foundation Σ_5 in that the functional operation is not restricted by postulation of the property P_0 and that the class $\mathfrak{N}$ is introduced. The property C_J for the class $\mathfrak{N}$ is a closure property, the class being closed as to convergence in the mean.

27. If a homogeneous fluid mass is rotating about an axis with a sufficiently small angular velocity there are two possible oblate spheroid figures of equilibrium, one nearly spherical, Σ_1, and one much flattened at the poles, Σ_2. If the fluid mass is compressible there are two figures of equilibrium nearly oblate spheroids. One approximates Σ_1 and is depressed in middle latitudes below the spheroid having the same polar and equatorial radii. The other approximates Σ_2 and is elevated in middle latitudes. The former theorem was proved by Airy, Callandreau and Darwin; the latter is a new theorem. This is established and a formula for the deviation from a true spheroid is obtained by Professor Moulton. He starts from an assumed relation between the pressure and density in the fluid and by making a compressibility parameter play a fundamental rôle is able to discuss the more flattened figure.

28. In his third paper Professor Carmichael shows how rational solutions of certain functional equations may be employed in solving problems of a certain class in the theory of diophantine analysis. In particular, several problems of Diophantus and Fermat are readily treated. The contents of both this paper and the preceding one by the same author will appear in his forthcoming "Introduction to Diophantine Analysis," to be published by Wiley and Sons.

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A GEOMETRIC DERIVATION OF A GENERAL FORMULA FOR THE SOUTHERLY DEVIATION OF FREELY FALLING BODIES.*

BY PROFESSOR WM. H. ROEVER.

(Read before the American Mathematical Society, October 25, 1913.)

WITHIN the last dozen years interest in the problem of the deviations of freely falling bodies seems to have been revived. There is a substantial agreement, among the writers who have treated this subject, as to the magnitude of the easterly deviation, their result being practically the same as that obtained

* See BULLETIN, vol. 20, No. 4, p. 175.