being a collector of specimens and found his chief outdoor recreation in the study of nature. He made two long canoe trips in the northwest of Canada. A carefully written diary illustrated with photographs of the second expedition which took him by rivers and lakes from Lake Superior to Hudson's Bay, is amongst the books which he left in his will to Columbia University.

He was president of the American Mathematical Society from 1894 to 1896, and served as lecturer on celestial mechanics in Columbia University from 1898 to 1901. The manuscript of his lectures shows that they must have cost him much labor; it contains long algebraic developments and is apparently intended to be a more or less complete account of the methods by which the motions of the moon and planets are calculated. His numerous honors include foreign membership in the Royal Society, the Paris Academy, and the Belgian Academy. He received the Schubert Prize (Petrograd), the Damoiseau Prize (Paris), the Gold Medal of the Royal Astronomical Society and in 1909 the Copley Medal of the Royal Society.

His chief characteristic was a single-minded devotion to the subject which he had made his own. A highly sensitive conscience was always apparent in his dealings with the world: one year he refused to accept the salary of his lectureship at Columbia because no students then appeared to attend the course, and this in spite of the fact that the endowment left him absolutely free to lecture or not as he chose. In later years, he rarely left West Nyack, owing to ill health. He died on April 16, 1914, from heart failure and was buried near the graves of his ancestors not far from his home.

E. W. Brown.

DICKSON'S LINEAR ALGEBRAS.

Linear Algebras. By L. E. Dickson, Ph.D. (No. 16, Cambridge Tracts in Mathematics and Mathematical Physics.) Cambridge, University Press, 1914. 8vo. viii + 73 pages.

> And still they read, and still the wonder grew,
> That one small tract contain so much. . . .

A substantial and systematic introduction to general linear algebras, associative and non-associative, a revision of Cartan's theory of linear associative algebras over the field of complex numbers, the results of Wedderburn's theory of such algebras over a general field, the relation of linear algebras to finite and infinite groups and bilinear forms, the consideration of various special algebras and a wealth of historical and bibliographical references in footnotes—in seventy-three pages!

The space at the author's disposal was doubtless limited. To have compressed so much material into it is indeed an achievement. With such an end in view the author could not afford to bandy words. He must needs make a point of being brief. And he has made this point well. His sentences are concise, and more often than not are condensed by the use of symbols rather than words or, now and then, by some ingenious form of brevity. The neatest of these is the use of (3_1), (8_2), etc., to denote "the first of the formulas (3)," "the second of the formulas (8)," etc. The equality sign is called upon to do duty in various rôles: thus, to denote the equivalence of two references, "*Proc. London Math. Soc.*, . . . (= *Coll. Math. Papers*, . . .)"; or to indicate the coordinates of a point, "the point $C = (c, d)$."

In attempting to attain to brevity, an author runs the danger of sacrificing clearness. Doubtless it is to avoid this pitfall that some authors spare no space in inserting every last bit of reasoning in their proofs and discussions. But a text-book from the pen of such an author affords the reader merely mental entertainment, whereas, in the reviewer's opinion, the ideal text-book on any except the elementary subjects should also require of the reader mental exercise. And often the more vigorous the exercise required, provided it remains within the possibilities of the reader, the more really lasting and beneficial is the entertainment.

Needless to say, the book under review is of the entertainment-plus-exercise type. The author demands continually of the reader that he supply bits of reasoning. But he is always very careful to make adequate suggestion as to their nature, usually by reference to previous sections and formulas. Thus actual clarity does not suffer; there is not an obscure place in the tract if one reads every word at its full value. Especially is the author to be complimented on his able revision of the complicated theory of Cartan in Part II.

For a complete understanding of a book, a reader needs not only to master the details but, as he reads, to strip them of their unessentials and assign them to their proper place in a skeleton structure of the whole subject, so that in the end he will see the aims, methods and facts, their significance and relative value, in a true perspective. Naturally he must do most of this digestive work himself, but the author may furnish him invaluable help by appropriate comments now and again. To do this, and do it well, is no easy task at the best. Still it is simpler, certainly, in a book of the type under review, in which the abstraction from the unessentials is already partly done, than in one of the purely entertainment type, in which such abstraction is rendered exceedingly difficult by the great mass of detail. But the present author makes no use of this advantage; in fact he makes next to no attempt to keep the reader properly oriented. As a result, his book presents always one and the same aspect, a desert of statement and proof with not a refreshing oasis in sight, where the reader may pause to rest and take account of stock. An amount of space, equivalent, say, to a half dozen unbroken pages, devoted to "oases" distributed judiciously throughout the book, would, in the reviewer's opinion, improve it greatly in raising the quality of clearness of the whole up to the standard of that attained in the details.

The tract is divided into four parts. Part I contains "definitions, concrete illustrations and important theorems capable of brief and elementary proof." It deals with the general linear algebra, associative and non-associative. Now the associative algebras of real interest are those possessing principal units, for in such an algebra without a principal unit division, if possible at all, is not unique. But with the introduction of non-associative algebras into the discussion, the presence or absence of a principal unit takes on new meaning. Hence in reading this first part, the beginner should keep an eye peeled for the appearance and disappearance not only of "associative," but also of "principal unit." In Part II we find the heart of the general theory, presented from Cartan's point of view for linear associative algebras over the field of complex numbers. The author chooses Cartan's treatment rather than those of Molien, Wedderburn, or Frobenius, because it alone remains within the bounds of linear algebras for its methods and proofs. The other treatments depend in whole or in part on the theory of groups or bilinear forms, and it is the purpose of the writer not to appeal to these to establish his results. The relations of these allied theories to linear algebras are adequately discussed, however, in Part III. Part IV contains the statement and elucidation of Wedderburn's results concerning linear algebras over an arbitrary number field. Numerous examples, illustrative of the theory, are given in smaller print throughout the book.

Part I begins with the development of the system of ordinary complex numbers. We are disappointed to find that the method of procedure here does not wholly conform to that used later in developing the system of general hypercomplex numbers. In this latter method the n-tuple form $(x_1, x_2, \cdots, x_n)$ of the hypercomplex number x is discarded just as soon as the processes, addition and multiplication by a scalar, requisite to obtain the more useful form $x = e_1x_1 + e_2x_2 + \cdots + e_nx_n$, where $e_1, e_2, \cdots, e_n$ are the n units, have been defined. But in the development of the ordinary complex numbers (and also in that of the matric algebra which follows in §§ 3, 4) there is no explicit mention made of multiplication by a scalar, and the couple form (a, b) of the complex number is retained almost to the end.

In § 2 a set of complex numbers is defined as a number field "if the rational operations can always be performed unambiguously within the set." The reviewer has racked his brain in a vain attempt to conceive a reason for the insertion of "unambiguously." If a rational operation on ordinary complex numbers—the definition can refer only to these, since no others have as yet been defined—is performable at all, i. e., if it is not division by zero, then the result is certainly unique.

The author adheres rigorously to the use of the concept of number field throughout. Thus, he defines (§ 5) as a linear algebra* over the field F the set of all numbers $\sum_{i=1}^{n} x_ie_i$, with coordinates x in F, subject to combination by suitably defined addition, subtraction, multiplication by a scalar in F, and by a distributive multiplication with the multiplication table $e_ie_j = \sum_{k=1}^{n} \gamma_{ijk}e_k$ $(i, j = 1, \cdots, n;$ γ's in $F)$.

In such an algebra, right-hand division, i. e., solution of $xx' = y$ $(y \neq 0)$ for x', is possible and unique if and only if the right-hand determinant of x, $\Delta(x) = \left| \sum_{i=1}^{n} x_i\gamma_{ijk} \right|$ $(j, k = 1, \cdots,$

* From now on in this review "algebra" will be used in the sense of "linear algebra."

n), is not zero. The left-hand determinant $\Delta'(x)$ is similarly related to left-hand division (§ 6). The number ϵ, if it exists such that $x\epsilon = \epsilon x = x$ for every number x of the algebra, the author sometimes calls "principal unit," but just as often "modulus," a term aptly and justly described by Study as "den auf alles Mögliche angewendeten Ausdruck 'Modul.'"* A necessary condition that there exist a principal unit is that both kinds of division be, in general, possible. The condition is proved sufficient if the algebra is associative, but for a non-associative algebra the author does not enlighten us (§ 7).†

In § 8 linear transformation of the units of an algebra is considered and the equivalence of two algebras under such linear transformation defined; in §§ 9, 10 it is noted that the general number x of an associative algebra is the root of an equation of degree $\leqq n + 1$ and satisfies any algebraic identity in an ordinary complex variable made up of functions without constant terms, or if the algebra moreover contains a principal unit, is a root of an equation of degree $\leqq n$ and satisfies *any* algebraic identity in an ordinary complex variable. By application of these results the author shows, in § 11, that every number of an associative algebra over the field $F(a)$ of real numbers, which has no nilfactors (divisors of zero), is a root of a quadratic equation, and hence obtains a neat proof that the only such algebras are $F(a)$ itself, the field $F(i)$ of ordinary complex numbers, and the algebra of real quaternions. In § 12 we find a discussion of the simplest properties of real quaternions, in § 13 a proof that the complex quaternion algebra and the complex matric algebra with four units are equivalent, though their real sub-algebras are not. In § 14 the author represents Cayley's eight unit algebra, a non-associative generalization of real quaternions, as a "quasi-binary algebra with real quaternion coordinates" and thus gives simple and pretty proofs that this algebra contains no nilfactors and that in it, too, the norm of a product equals the product of the norms.

We now return to the general theory and introduce the right and left-hand characteristic determinants and equations $\delta(\omega)$, $\delta'(\omega)$, $\delta(\omega) = 0$, $\delta'(\omega) = 0$, where $\delta(\omega)$ and $\delta'(\omega)$ are derived from $\Delta(x)$ and $\Delta'(x)$ by subtracting ω from each

* Encyklopädie der math. Wiss., I, A, 4, p. 162.

† It is a simple matter to construct an example to show that the condition is not sufficient for a non-associative algebra.

element of the principal diagonals. The general number of an associative algebra is a root of both $\delta(\omega) = 0$ and $\delta'(\omega) = 0$. Suitable variations of this theorem for an algebra with a principal unit and for an arbitrary algebra are given. In § 17 a function of the x_i and the γ_{ijk} which reproduces itself under an arbitrary linear transformation of units with a factor depending only on the coefficients of the transformation is called a covariant (invariant, if it involves only the γ's). The determinants $\Delta(x)$, $\Delta'(x)$, $\delta(\omega)$, $\delta'(\omega)$ are absolute covariants of the general algebra. The proof given in § 16 of this is incomplete,* since the statement "let $\Delta(x)$ be not identically zero, so that there exists a modulus" assumes the algebra associative and hence leaves the theorem unproved for non-associative algebras for which $\Delta(x) \neq 0$. The theorem is applied in § 18 to obtain the three (two) types of non-equivalent binary algebras over $F(a)[F(i)]$ with a principal unit.

The rank r of an associative algebra is the least positive integer such that x^r is a linear combination of lower powers of x whose coefficients are rational functions of $x_1, \cdots, x_n$ with coefficients in F. The corresponding equation $R(x) = 0$ is called the rank equation. If the algebra has a principal unit, $R(\omega) = 0$, $\delta(\omega) = 0$, $\delta'(\omega) = 0$ all have the same distinct roots. As an application, the author gives Study's classification of complex ternary associative algebras with a principal unit into five non-equivalent types according to the five possibilities for the degree and the multiplicity of the roots of the rank equation (§ 20).

The author defines an associative algebra with n units and a principal unit as reducible in its field F if it contains n numbers $e_1, \cdots, e_p; E_1, \cdots, E_q$, linearly independent with respect to F,

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* The author's proof goes astray in assuming the existence of a principal unit. But the only use he makes of this assumption is to show, after he has proved that $D(X) = c\Delta(x)$, where c depends only on the coefficients c_{ij} of the transformation of units, that $c = 1$. Now, by inspection, c is the quotient of a homogeneous expression of degree $2n$ in the c_{ij} by $|c_{ij}|^2$. Since c cannot vanish unless $|c_{ij}|$ vanishes—otherwise we could exhibit values for the x_i, γ_{ijk} for which $\Delta(x) \neq 0$, while $D(X) = 0$—and $|c_{ij}|$ is irreducible, $c = k|c_{ij}|^l$, where k is independent of the c_{ij}. But evidently $l = 0$, and, by applying the identical transformation of units, $k = 1$.

Professor Dickson's proof holds for any algebra with a principal unit. In a letter to the reviewer he points out that the case of an algebra without a principal unit may be treated by enlarging it to an algebra with a principal unit by adjoining a new unit (as in the proof of Theorem 3, § 15), and also calls attention to a later proof by Miss Hazlett, *Annals of Mathematics*, vol. 16 (1914), p. 2.

such that $e_iE_j = E_je_i = 0$, $(i = 1, \cdots, p;\ j = 1, \cdots, q)$, and proves that such an algebra is the direct sum of two algebras.* He then sets up Scheffer's criterion for reducibility. In § 22 the direct product of any two algebras is defined and Wedderburn's generalization of Scheffer's theorem identifying an associative algebra, which has the quaternion algebra as a subalgebra and its principal unit as principal unit, as the direct product of the quaternion algebra and a second algebra.

In the last main paragraph (§ 23) of Part I a set of normalized units for any algebra is developed. Since their main purpose is to form a point of departure for Part II, we have need to describe them here only for an associative algebra with a principal unit ϵ. Suppose that the solutions of $\delta(\omega) = 0$ for a particular number a of the algebra are $\omega_1, \cdots, \omega_h$ of multiplicities $m_1, \cdots, m_h$, and consider one of these solutions, say ω_1. Now $a - \omega_1\epsilon$ is a nilfactor, that is, there exist solutions $y \neq 0$ of $(a - \omega_1\epsilon)y = 0$ or $ay = \omega_1 y$; take a complete system of solutions y, say t in number, as the first t of the new units $e_1, e_2, \cdots$. If $t < m_1$, there exist solutions z of $az = \omega_1 z + \sum_{i=1}^{t} c_ie_i$; take a complete system of solutions z, say t_1 in number, for the next t_1 of the new units. If $t + t_1 < m_1$, repeat; we will finally obtain m_1 numbers $\alpha_1, \alpha_1', \cdots, \alpha_1^{(m_1-1)}$, such that $(a - \omega_1\epsilon)\alpha_1^{(i)}$ is a linear function of the preceding α_1's. Similarly, there exist, corresponding to the root ω_j, m_j numbers $\alpha_j, \alpha_j', \cdots, \alpha_j^{(m_j-1)}$, such that $(a - \omega_j\epsilon)\alpha_j^{(i)}$ is a linear function of the preceding α_j's. The $n = m_1 + m_2 + \cdots + m_h$ new units thus obtained are linearly dependent; they are termed a set of units normalized relatively to the number a.

Part II consists of a " revision of Cartan's general theory of complex linear associative algebras with a modulus." In § 25 units having a character are developed. Denote the principal unit, expressed in terms of the units of § 23, by $\epsilon = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_h$, where ϵ_i is a linear combination of the α_i's. (The ϵ_i are called the partial moduli.) Then the number η is said to have the character (j, k) if $\epsilon_j\eta = \eta$, $\eta\epsilon_k = \eta$, and every other combination of η with an ϵ_i by multiplication is zero. Now $\epsilon_j\alpha_j^{(i)} = \alpha_j^{(i)}$, $\epsilon_k\alpha_j^{(i)} = 0$ and in particular $\epsilon_i^2 = \epsilon_i$,

* He might have generalized the definition by demanding, instead of the existence of a principal unit, the existence of two numbers e, E, where e (E) is a linear function of the $e_i(E_i)$ with coefficients in F, which is not a nilfactor with respect to all other such linear functions, and the proof would still hold with but slight variation.

$\epsilon_i \epsilon_j = 0$; furthermore there exist m_j linearly independent linear combinations of the $\alpha_j^{(i)}$, such that the product of any one of them into a certain ϵ_i gives itself, and its product into any other ϵ_i gives zero, i. e., such that they have definite characters $(i, \cdot)$. Hence we have obtained n new units $\epsilon_1, \epsilon_2, \cdots, \epsilon_h, \eta_1, \cdots, \eta_k$ each having a definite character. We wish further to show that we can choose the η's so that each is nilpotent, i. e., so that some power of it is zero. Now the square of every η of character (i, j), $i \neq j$, is zero, since the product of a number of character (i, j) by a number of character (k, l) is zero, if $j \neq k$. And since the sum or product of two numbers of like character, if not zero, is of that character, all the numbers of character (i, i) of the algebra form a sub-algebra Σ_i; it is possible to choose the units of this sub-algebra, other than ϵ_i, as nilpotent numbers. Thus we have arrived at a set of normalized units consisting of the partial moduli and $n - h$ nilpotent numbers each having a definite character. We shall refer to such a set of normalized units as being of type K.

We now separate algebras into two categories, according as the determinant $\Delta(x)$ does not or does contain a non-linear irreducible factor. To prove the principal theorem for algebras A_1 of the first category, i. e., that for such an algebra a set of units of type K can be found having the further property that $\eta_i \eta_j$ is a linear function of those η's whose subscripts exceed i and j and have the same character as $\eta_i \eta_j$ (property L), the author shows that the nilpotent numbers of A_1 form a sub-algebra N, for which units η of definite characters enjoying the property L can be found. Conversely, if an algebra is given possessing a set of units of type K enjoying the property L, the γ_{ijk} of the multiplication table are determined (except for the non-vanishing ones in the $\eta_i \eta_j$) and inspection shows that $\delta(\omega) = (x_1 - \omega)^{k_1} \cdots (x_h - \omega)^{k_h}$, that is, that the algebra is of the first category. An algebra such as N is called nilpotent.

To every irreducible factor of the characteristic determinant for an algebra A_2 of the second category corresponds a sub-algebra of A_2, just as in the case of an algebra A_1. But here at least one of these irreducible factors is not linear. Each sub-algebra corresponding to such an irreducible factor contains linear combinations of nilpotent numbers which are not nilpotent. These numbers are peculiar to algebras A_2

and form the basis of Cartan's treatment of them. The details of the treatment are too intricate to permit of an adequate description in a brief space. We content ourselves with a statement of the principal results. First, any algebra A_2 contains a sub-algebra equivalent to a matric algebra with more than one unit. Hence Cartan's classification of algebras into those of the first and second categories coincides with that of Scheffers into those without and with quaternion sub-algebras. As the final result we have a process by means of which the algebras A_2 may be derived from the algebras A_1: the general number of an A_2 is obtained from the general number of an A_1, expressed in terms of units of type K having the property L, "by regarding the coefficient of ϵ_i to be a square matrix of p_i^2 elements and that of η_l, of character (i, j), to be a rectangular matrix of p_i rows and p_j columns, these matrices to be regarded as commutative with each ϵ and η." Thus we obtain from the units of an A_1, the multiplication table and the characteristic determinant corresponding to them, the units, multiplication table and characteristic determinant of an A_2.

In §§ 49, 50 the composition of an algebra with respect to the presence or absence of invariant sub-algebras is discussed. A sub-algebra I of an algebra A is termed invariant if the product of any number of I and any number of A in either order is a number of I. An algebra having no invariant sub-algebra is called simple and one having no nilpotent invariant sub-algebra semi-simple. A general algebra, A_1 or A_2, is the sum, but not necessarily the direct sum, of a semi-simple algebra and a nilpotent invariant sub-algebra. A semi-simple algebra, if not simple, is the direct sum of simple algebras and conversely. Finally, a simple algebra is matric, unary if an A_1, non-unary if an A_2, and conversely.

In concluding Part II the author shows that a commutative algebra must be an A_1 and, in particular, an A_1 which is the direct sum of its sub-algebras $\Sigma_1, \cdots, \Sigma_h$.

Part III begins with a consideration of the correspondence between associative algebras and linear groups. To each number y of any algebra correspond two linear homogeneous transformations (y), $[y]$ of the variables $x_1, \cdots, x_n$ into the variables $x_1', \cdots, x_n'$, defined by the equations obtained from the products $x' = xy$, $x' = yx$. If the algebra is associative, the set of transformations (y) is closed and so is the set of transformations $[y]$. If further the algebra contains a principal unit, there corresponds to it in each set the identical transformation, and hence the inverse of any transformation corresponding to a non-nilfactor exists (and belongs to the same set). Thus the two sets of transformations (y), $[y]$, corresponding to the non-nilfactors y, form groups G, G'. These groups are simply transitive, mutually commutative (reciprocal in the sense of Scheffers), and each is its own parameter group. Conversely, any two simply transitive reciprocal groups of linear transformations can be transformed by the same change of variables into a pair of groups G, G', defined by an associative algebra with a principal unit. A classical example of the representation of linear transformations by hypercomplex numbers is then given: if q and Q are variable and q_1, q_2 given quaternions, then $Q = q_1\, q\, q_2$ represents the 7-parameter group of transformations of similitude in R_4 leaving the origin invariant; the group represented by the equation when, in particular, q_1 and q_2 are conjugate and q and Q vectorial quaternions, is not the corresponding group in R_3, as the author states, since it does not contain the stretchings from the origin but only the rotations about it.

Section 53 shows that the theory of bilinear forms is equivalent to that of matrices and also, in the last analysis, to that of associative algebras; section 54 explains the relations between finite groups and their group matrices to linear algebras; and section 55 treats of Dedekind's idea of considering the relations giving the values of the products e_ie_j $(i, j = 1, \cdots, n)$ of the units of a commutative associative algebra as ordinary algebraic equations in the units as unknowns.

The decomposition of a linear associative algebra over a field F within this field is precisely the same as that given above (§ 50) for a complex associative algebra within the field $F(i)$, except that instead of the last statement we now have: any simple algebra over F is the direct product of a division algebra and a simple matric algebra, each over F, and conversely. A division algebra is one without nilfactors. The relation of the particular case, $F = F(i)$, to the general one is made clear by the fact that the only division algebra over $F(i)$ is $F(i)$ itself. Further, if F is the field $F(a)$ of all reals, the only division algebras, besides $F(a)$, are $F(i)$ and the algebra of real quaternions. Now, since it can be shown that a commutative matric algebra has a single unit, it follows that any real commutative algebra without nilpotent numbers is the direct sum of unary algebras equivalent to $F(a)$ and binary algebras equivalent to $F(i)$. Weierstrass showed that a real commutative associative algebra, possessing a principal unit and such that the only algebraic equations with an infinitude of the numbers of the algebra as roots are those in which the coefficients are multiples of one nilfactor, is equivalent within $F(a)$ to just such a sum of real unary and binary algebras and conversely.

In §§ 58–60 are given examples of various types of division algebras previously published by the author. In § 61 results concerning difference algebras, the composition, difference and reduction series of an algebra are given, and, in particular, Wedderburn's theorem to the effect that an associative algebra over F can be decomposed in one and but one way into the direct sum of irreducible algebras with principal units and, if the given algebra has not a principal unit, an algebra without a principal unit. The closing section (§ 62) of the tract contains brief mention of the work of Berloty, Scheffers, and Hausdorff on analytic functions of hypercomplex numbers.

Attention is called to the following misprints: page 4, line 8, read *of* instead of *by*; page 24, last line, $\xi^2-2L\xi$ instead of $\xi^2+2L\xi$; page 32, last line, e_5 instead of e; § 26, last line, $e_3\epsilon_1$ instead of $e_3\epsilon_4$; § 33, third line, $-\omega^5$ instead of ω^5; § 33, fifth line, omit "the" before "normalized units"; page 43, "(71)" applies to both equations; page 48, ninth line from bottom, insert "variables" before [2, 2]. In § 6, the condition $y \neq 0$ is omitted; on page 41, in the statement of the lemma, $u \neq 0$, and in § 57, $k \neq 0$. In light of the complexity of the subscript notation needed, it is surprising that there are so few typographical errors. As the author says, in closing the introduction, "the quality of the printing speaks for itself." There is no index, but the table of contents is so detailed that it practically takes the place of one.

We have already pointed out the lack of general comment; in this connection we wish now to note the meagreness of the treatment accorded to nilfactors. We should like to have seen such an important concept introduced at the first opportunity, namely in § 4, by calling attention to the fact that the algebras given as examples differ essentially, in that the matric algebra contains nilfactors whereas $F(i)$ does not. And it would be natural for us to introduce the definition of nilfactors in § 6 in connection with $\Delta(x)$ and $\Delta'(x)$, and point out that they are simply the numbers for which these determinants vanish. As a matter of fact, nilfactors are not defined until § 19 and then only for an associative algebra with a principal unit ϵ, whereas the concept may be extended to any algebra; we learn there that $x - \omega\epsilon$ is a nilfactor if and only if ω is a root of the rank equation. This is the first and last mention of " nilfactor " in Parts I, II. It is not even pointed out that the nilpotent number is a special type of nilfactor. We learn that a necessary and sufficient condition that a number be nilpotent is that every root of its characteristic equation be zero, but nowhere are we told that the corresponding condition for a nilfactor is that one root of its characteristic equation vanish.

The conditions on the γ_{ijk}, that multiplication be associative, all but escape mention; we find them buried in a proof on page 59. If the author was to accord them a place at all—as he certainly should—should it not be their natural one in § 5, where multiplication is defined and discussed? Finally, the following points did escape mention: an associative algebra with n units with (without) a principal unit, which is of rank $n(n+1)$, is commutative; a binary algebra with a principal unit is associative and commutative (essential to the complete understanding of § 18).

Though we have had differences with the author, we are in the main at one with him. He has written an able and comprehensive book on an abstruse subject, and a book which satisfies a long felt need, as an introduction and also as an up-to-date book of reference. It should prove invaluable to the beginner and advanced student alike.

W. C. GRAUSTEIN.