and the points of S', a one-to-one reciprocal correspondence preserving limits.*

The following theorems may be easily established with the assistance of Theorem 18 of § 2 and Theorem IV of my paper "On a set of postulates which suffice to define a number-plane."†

THEOREM A. *Every two-dimensional space that satisfies Hilbert's plane axioms of Groups I and II (or Veblen's I–VIII) together with Axiom A is equivalent, from the standpoint of analysis situs, either to a two-dimensional euclidean space or to an everywhere dense subset thereof.*

THEOREM B.‡ *Every two-dimensional space that satisfies Hilbert's plane axioms of Groups I, II and III (or Veblen's I–VIII, XII) together with Desargues' theorem and Axiom A is descriptively equivalent either to a two-dimensional euclidean space or an everywhere dense subset thereof.*

COROLLARY. *Pascal's theorem§ is a consequence of Hilbert's plane axioms of Groups I, II and III together with Desargues' theorem and Axiom A.*

THE UNIVERSITY OF PENNSYLVANIA.

A TYPE OF SINGULAR POINTS FOR A TRANSFORMATION OF THREE VARIABLES.

BY DR. W. V. LOVITT.

(Read before the American Mathematical Society, December 31, 1915.)

IN the *Transactions* for October, 1915, I discussed some singularities of a point transformation in three variables

$$(1)\qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

* The statement that such a correspondence preserves limits signifies that if A is a *point* of S, M is a *point set* of S, and A' and M' respectively are the corresponding point and point set of S' then P is a *limit point* of M if, and only if, P' is a limit point of M'. Here P is said to be a *limit point* of M if, and only if, every *triangle* of S that contains P *within* it contains *within* it at least one *point* of M distinct from P.

† *Transactions of the American Mathematical Society*, vol. 16 (1915), pp. 27–32.

‡ For a corresponding theorem regarding Axiom B (cf. footnote in § 2) see Vahlen, loc. cit., pp. 158–163.

§ Cf. Hilbert, loc. cit., p. 40.

with determinant

$$J(u, v, w) = \begin{vmatrix} \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix}.$$

In that paper the functions ϕ, ψ, χ were not necessarily analytic but it was presupposed that

(a) the functions ϕ, ψ, χ are of class C'''* in a neighborhood of the origin $(u, v, w) = (0, 0, 0)$;

(b) the following initial conditions are satisfied:

$$\phi(0, 0, 0) = \psi(0, 0, 0) = \chi(0, 0, 0) = 0;$$

(c) $J(0, 0, 0) = 0$;

(d) at the origin $(u, v, w) = (0, 0, 0)$ at least one of the determinants of the matrix

$$\begin{matrix} J_u & J_v & J_w \\ \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{matrix}$$

is different from zero.

In the present note I desire to show that the results of that paper apply to a transformation of the form

$$\begin{aligned} &f(x, y, z;\ u, v, w) = 0, \\ (2)\qquad &g(x, y, z;\ u, v, w) = 0, \\ &h(x, y, z;\ u, v, w) = 0. \end{aligned}$$

The functions f, g, h are not necessarily analytic but it will be presupposed that

(a') the functions f, g, h are of class C''' in a neighborhood of the origin $(x, y, z;\ u, v, w) = (0, 0, 0;\ 0, 0, 0)$;

(b') the following initial conditions are satisfied:

$$f(0, 0, 0; 0, 0, 0) = g(0, 0, 0; 0, 0, 0) = h(0, 0, 0; 0, 0, 0) = 0;$$

(c') $B \equiv \dfrac{\partial(f, g, h)}{\partial(u, v, w)} = 0$, at the origin;

(d') at the origin $(x, y, z;\ u, v, w) = (0, 0, 0;\ 0, 0, 0)$ at least one of the determinants of the matrix

* We shall say that a single-valued function f of (u, v, w) is of class C''' if $f(u, v, w)$ and its partial derivatives of orders one, two, and three are continuous in a region in which f is defined.

$$\begin{matrix} B_u & B_v & B_w \\ f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{matrix}$$

is different from zero;

(e') $\dfrac{\partial(f, g, h)}{\partial(x, y, z)} \neq 0$, at the origin.

Assumption (e') assures us of the existence of a solution of equations (2) of the form of equations (1). We shall now consider that equations (1) have been obtained from (2), and proceed to show that, on account of the conditions (a'), (b'), (c'), (d'), (e'), equations (1) satisfy the conditions (a), (b), (c), (d).

Assumptions (a) and (b) follow at once from (a') and (b') as a result of the ordinary theorems on implicit functions.* From the equation

$$\frac{\partial(f, g, h)}{\partial(x, y, z)} \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial(f, g, h)}{\partial(u, v, w)}, \tag{3}$$

on account of our assumptions (c') and (e'), we find

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 0, \text{ at the origin.}$$

The left hand member of the last equation is the jacobian J of the equations (1) and hence the condition (c) is satisfied. From equation (3) it follows, on account of (e') and (c), that if not all of B_u, B_v, B_w, then also not all of J_u, J_v, J_w, vanish at the point in question. It is easily verified that if the determinant B is of rank two, then also is the determinant J of rank two. Hence it follows from assumption (d') that the assumption (d) is satisfied.

That the determinant J is of rank two is seen as follows. Let α, β stand for some two of the variables u, v, w. Then

$$\begin{aligned} f_x\phi_\alpha + f_y\psi_\alpha + f_z\chi_\alpha &= -f_\alpha, \\ f_x\phi_\beta + f_y\psi_\beta + f_z\chi_\beta &= -f_\beta, \\ g_x\phi_\alpha + g_y\psi_\alpha + g_z\chi_\alpha &= -g_\alpha, \\ g_x\phi_\beta + g_y\psi_\beta + g_z\chi_\beta &= -g_\beta. \end{aligned}$$

* See Bliss, Princeton Colloquium Lectures, p. 8.

Whence

$$f_\alpha g_\beta - f_\beta g_\alpha = \frac{\partial(\phi\psi)}{\partial(\alpha\beta)} f_x g_y + \frac{\partial(\phi\chi)}{\partial(\alpha\beta)} f_x g_z + \frac{\partial(\psi\phi)}{\partial(\alpha\beta)} f_y g_x + \frac{\partial(\psi\chi)}{\partial(\alpha\beta)} f_y g_z.$$

Now if the determinant B is of rank two we must have

$$f_\alpha g_\beta - f_\beta g_\alpha \neq 0, \text{ at the origin;}$$

whence it follows that some one of the functional determinants appearing on the right hand side of the last equality does not vanish at the origin, and hence J is of rank two.

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THE HISTORY OF THE CONSTRUCTION OF THE REGULAR POLYGON OF SEVENTEEN SIDES.

Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks. Eine historisch-kritische Darstellung. Von R. GOLDENRING. Leipzig und Berlin, Teubner, 1915. 8vo. 6+69 pp. Price 2.80 marks.

TILL near the close of the eighteenth century, mathematicians felt sure that the only regular polygons which could be constructed with ruler and compasses were those known to the Greeks. But the extraordinary discoveries of Gauss, while yet in his teens, greatly extended this class of polygons and settled for all time the limits of possibilities for such constructions. In this connection the discovery that the regular polygon of seventeen sides could be constructed with ruler and compasses was not only one of which Gauss was vastly proud throughout his life, but also, according to Sartorius von Waltershausen,* one which decided him to dedicate his life to the study of mathematics. Two of Gauss's notes recording this turning point of his career have been preserved. The very first entry in his "Wissenschaftliches Tagebuch 1796–1814" † is: "Principia quibus innititur sectio circuli ac divisibilitas eiusdem geometrica in septemdecim partes etc. *Mart.* 30. *Brunsvigae.*" And again, in his own copy of his Disquisitiones

* Gauss zum Gedächtniss, Leipzig, 1856, p. 16.

† This was "mit Anmerkungen herausgegeben von F. Klein," *Math. Annalen*, Band 57 (1903), pp. 1–34.