author had a thorough knowledge of the work of Euclid and that he made use of it with remarkable ease. He is therefore a circle squarer of a type different from those met by De Morgan, whose common characteristic is a complete and stubborn ignorance of the science of geometry.

Porta on the contrary knew the classics of the science and was so imbued by the methods of the ancients as to make an evident effort to imitate their procedure and their style. And finally, if we take account of the fact that he lived at a time when geometricians could not walk without the steady support of Euclid and Archimedes, to whose invincible spell they had to submit, we conclude that De Morgan's judgment of Porta is too severe. If indeed we consider him in the large group of authors of paradoxes studied with so much care by the illustrious professor of the University of London, Porta appears—I will not say as a new Archimedes, but at least a monoculus in terra cœcorum.

UNIVERSITY OF GENOA,
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## PIERPONT'S FUNCTIONS OF A COMPLEX VARIABLE.

*Functions of a Complex Variable.* By JAMES PIERPONT. Ginn and Company, Boston, 1914. xiv + 583 pp. 8vo. Price $5.00.

OUT of the general development of higher analysis certain portions have gradually come to be recognized as suitable for introductory material and designated as Theory of Functions, or more precisely, as Functions of a Complex Variable. We have also Functions of a Real Variable, and it might seem as if these should come first; but there is a certain simplicity and elegance about functions of a complex variable that makes their study especially attractive.

In this subject there is much that has been unsettled as to subject matter and method of treatment, and many of the books have quickly become out of date, or have lacked the accuracy now possible. It is therefore a matter of congratulation when a book of this kind comes from the interesting pen of Professor Pierpont. Such a book is bound to have a marked influence on the development of an ideal course.

This book is intended for students who have only completed the usual work in the calculus, and especially for those who do not intend to specialize in mathematics. Doubtless no one would be expected to cover all of these nearly 600 pages, but the subjects treated are all suitable for a first course, and those who have studied parts of the book will always find it useful for reference.

The teacher will notice at once the entire absence of problems to be solved. Much of the algebraic work, however, is quite condensed, and any one who masters this will have plenty of exercise, and with results better coördinated with the text than the solution of independent examples would give.*

The author deals gently with our reasoning powers, though he does not hesitate to tax our skill in the manipulation of formulas, and so perhaps the text which he produces is well adapted to the state of development of those who will use it. There seems to be a tendency in the schools to cultivate technical skill and to slight the theoretical side of mathematics. It is undoubtedly easier to train men to accurate mathematical work, and this training is of course very valuable; but it is more important to understand a subject than merely to be able to use it, and it might be well on account of this tendency to emphasize the theory more than we otherwise would.

In particular, the theory of irrational numbers forms the basis of all analysis, and is fundamental even in parts of elementary geometry and algebra. It is the opinion of many mathematicians that this theory cannot be taught in the high school, nor even to college freshmen; but it is certainly necessary to any rational treatment of limits and series, and one would think that it might be taken up with the calculus. Yet here is a book for students who have completed one or two years in the calculus, who are supposed to have some knowledge of series and a considerable knowledge of the properties of integrals, but who are still too immature to comprehend the Dedekind theory, or to prove, for example,

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* A good collection of examples may be found in the English translation of Burkhardt's Einführung, etc., referred to by our author on page 3: Theory of Functions of a Complex Variable, H. Burkhardt, translated by S. E. Rasor, D. C. Heath and Co., Boston, 1913. There are examples also in Forsyth's Theory of Functions of a Complex Variable, Cambridge University Press, 2nd ed., 1900, and in Principes de la Théorie des Fonctions Elliptiques et Applications, by P. Appell and E. Lacour, Paris, 1897.

that a variable always increasing and always less than some fixed number will have a limit (page 29). The author explains his view very clearly on page 33, but we think that this subject can be made quite simple, so much of it as is necessary for the proofs of a few fundamental theorems, and that it is well worth the time that might have to be taken from more advanced technical study. Even students who do not go on with mathematics will know better the mathematics that they do get, and will be better able to use it, if they understand better the foundations on which it rests.

Extreme care is shown in the accuracy of the proofs, yet there are certain forms of expression which might lead the immature student into loose ideas as to the necessity of always having a proof. The words "obvious" and "obviously" can usually be cut out without any loss whatever, and when they occur they are apt to leave the impression that some things are to be taken in mathematics as obvious. We find that "the theorem mentioned above is self-evident and requires no proof," that the reader "knows" some things (page 7), and that the discussion given of the law of the mean (page 143), although not an analytic proof, will make the reader "feel in the most convincing manner" that the law is true. Sometimes also a theorem is true because "the figure shows it" (see, for example, pages 58 and 139). The student should clearly understand that such considerations can form no part of a logical system.

These remarks do not apply to the work as a whole, but only to a few sections. In general, the importance of rigorous proofs and accurate details is emphasized.

The author has an informal way of discussing many things that makes them very interesting. Instead of a collection of dry facts and formulas we see the subject as it develops. See, for example, his remarks on imaginary numbers (page 10), the treatment of the $\epsilon$-notation (pages 32–34 and 132), differentiation and integration at the beginning of Chapter VI, the significance of Cauchy's integral theorems (page 215), Taylor's theorem (page 223), the $\sigma$-function (page 359), and the $\vartheta$-functions with zero arguments (page 431).

The book begins with a historical sketch of two pages. We should like to have seen a fuller account of the early history of this subject. A few historical remarks scattered through the book give a hint as to how interesting such an account would be (see pages 23, 91, 289, 300, 395, 402, 410, 412, 418, 423, 425–426, 454–455). The names of Abel, Cauchy, Gauss, Jacobi, Legendre, and Weierstrass occur frequently, and several others are mentioned, but these names would mean much more if the student knew something of the men. In only a single case (pages 395 and 401) is there a reference to any of their writings, and in the entire book there are only seven specific references to other books or publications.

A much fuller index and a table of formulas would have been very useful, and the number of cross references could have been increased to great advantage.

The first chapter gives the usual account of the representation of complex numbers by points in a plane. There is no mention in the book of the linear function except once on page 88, nor of Riemann surfaces. We are accustomed to the early introduction of these topics, but the author has been able to do without them, and regards other subjects as more important for students who do not intend to specialize in mathematics.

Two chapters deal respectively with real and complex series, but the first of these two chapters is concerned chiefly with questions of convergence, while Chapter III takes up the various properties of series, operations with series, power series, and double series. The author explains very carefully the notion of limits (page 33), and introduces the admirable notation so freely used in his Functions of a Real Variable,

$$\text{“}\epsilon > 0,\ m,\ |\, c - c_n \,| < \epsilon,\ n > m.\text{”}$$

We should like to call attention also to his excellent treatment of the "associative and commutative" properties of series (pages 64–71), and of "row and column series" (pages 80–84). One theorem on the removal of parentheses (the second case at the bottom of page 67) might be made a little more general by assuming, not that $A$ is a positive term series, but that the terms in the parentheses are all positive while the parentheses may have either sign. This, indeed, is the theorem required on page 70.

These two chapters are followed by a chapter on the functions employed in elementary mathematics. A third of this chapter (about 13 pages) comprises all that we have on algebraic functions, the book being devoted almost entirely to one-valued or uniform functions. The rest of the chapter is a study of the exponential, circular, and logarithmic functions, starting from their series developments.

Chapter V is on real variables, being a résumé of some parts of the calculus and an account of curvilinear and surface integrals. It is chiefly in this chapter that the author appeals to intuition, or bases his theorems on geometrical considerations that do not constitute proofs. As we have already explained we think that a more rigorous treatment would not have been so very much more difficult. There are some interesting physical applications at the end of the chapter, to work, potential, electric current, and Stokes's theorem.*

Chapter V prepares the way for the study of complex differentiation and integration in Chapter VI. The Cauchy-Riemann equations are given, the theorem on conformal representation by means of a function having a derivative, and the properties of integrals of such functions. Steady (that is, uniform) convergence is also taken up, and the integration and differentiation of series. The treatment of all these subjects is very clear and simple.

Then we have a chapter on analytic functions, with Cauchy's theorems, Taylor's theorem, Laurent's theorem, and Fourier's development; a chapter on infinite products leading to Weierstrass's theorem (Mittag-Leffler's theorem not being mentioned); and a chapter applying these theorems to the study of the Beta and Gamma functions, and giving a somewhat difficult account of asymptotic expansions. The subject of analytic continuation as presented in the first of these three chapters is interesting. This term is used here in a slightly broader sense than usual; namely, to denote the process of finding the value of an analytic function at a point $z$ when its values are known along some piece of a curve or line (page 225). In the chapter on infinite products† the sine

* On page 156 the set of equations immediately following (5) is parenthetical, put in for the purpose of deriving formulas to be applied to (5). It would have been clearer if these equations had been put in as a footnote, or printed in small type.

† On page 267 an infinite product is said to be convergent when the limit of the product of the first $n$ factors is finite and not zero, or "when one of the factors is zero." This is not the usual definition and with this definition the theorems which follow are not all true.

On page 280, in considering the associative character of infinite products the author omits entirely the question of removing parentheses, which he treats very carefully in the case of infinite series (page 67).

and cosine products and associated series are given. It is always a delight to the student to find so many new and wonderful properties of the long familiar functions of trigonometry.

The relation of the modern function theory to the study of elliptic functions is about the same as the relation of analytic geometry to the study of conic sections. There are three chapters (X–XII) on the elliptic functions, the first on the functions of Weierstrass, the second on the functions of Legendre and Jacobi, and the third on the $\vartheta$-functions and the relations of the two systems of the preceding chapters. As usual the functions of Weierstrass are developed from the point of view of their periodicity and the functions of Legendre and Jacobi from the elliptic integrals. There is a certain amount of duplication in the two systems, but both are needed, one being better adapted to some applications and the other to other applications. The $\vartheta$-functions are the simplest functions in the older system and it is often convenient to express the other functions in terms of them. They correspond to the $\sigma$-functions of Weierstrass, and the exact relation between the two systems is obtained from the study of these two simplest types. We have in these three chapters a very good introduction to the elliptic functions with plenty of detail to work out and one or two applications. More applications would have been welcome.

Finally, three chapters (XIII–XV) on linear differential equations introduce us to some of the functions defined by equations of the second order. This subject is not usually considered in the text-books on the theory of functions, but these chapters put into our possession some of the functions which are most important in mathematical physics, and the methods used are only those developed in the preceding pages. The author's course in writing these chapters is in line with the practice of some writers on the calculus to add a chapter on the solution of differential equations as furnishing many valuable results from the methods previously studied, as well as the best possible kind of practice in these methods.

One or two details may be noticed. It would probably have conduced to clearness if the author on pages 41 and 284 had referred to the extended law of the mean or Taylor's theorem in finite form, instead of simply "the law of the mean" as he does. The proof at the bottom of page 245 requires that $c \neq 0$. For $c = 0$ the treatment would be somewhat different. On page 344, an elliptic function that has no pole in a parallelogram of periods is a constant, not because it "has no singular point anywhere in the infinite plane," but because it is "less than some $G$" in the parallelogram, and then less than this $G$ everywhere in the plane. On page 364 (9), the coefficient of $z^9$ should be $-s_6/8 + s_2^2/32$. This shows that the law of the series is not as simple as the second and third terms would indicate. On page 429, line 1, we cannot get $a_0$ by putting $q = 0$ until we have shown that $a_0$ does not depend on $q$. The terms of the sine-series for $\vartheta_1(v)$ do not vanish when $v = m + n\omega$, as stated at the top of page 430, for $n \neq 0$. On page 473, if $m = l + \frac{1}{2}$ and $r = -m$, $c_{2l+1}$ will not vanish but will be undetermined, and all the $c$'s of odd index after this $c$ will contain it as a factor. We can, however, arbitrarily give to this $c$ the value zero and so get an integral independent of $y_1$ and without logarithms as desired. On pages 481–483 the formulas are not quite consistent as to the sign-factor $(-1)^m$. The matter may be straightened out by putting this factor into equation (2) and taking it out of equation (3) and out of the expression for $\partial U_1/\partial s$. In equation (6) $m$ should be used instead of $n$ and $2^m\Pi(m)$ should be in the numerator and not in the denominator. In the expression for $\partial V_1/\partial s$ the first term should contain the factor $2^m$, and the second term should be $C\, 2^{m-1}\Pi(m)\omega(m)J_m(x)$ instead of "$1/2\ C\ J_m(x)$." Also $\omega(0)$ should be put equal to 0 and not equal to 1. In (7) the logarithmic term should have the factor 2, "$\Pi(m)$" should be $\Pi(k)$, and at the end there should be the factor $(x/2)^{2k}$. On page 551, eight lines from the bottom the factor $|\gamma/a|^{s+1}$ should be $|t/a|^{s+1}$, and $H/x^{s+1}$ should be $Ht^{s+1}/x^{s+1}$; in the integral in (10) $t^{\lambda-1}$ should be $t^{\lambda-s}$, and the limit of the integral will be $\Gamma(\lambda + s - 1)$.

The criticisms which we have offered do not reflect on the validity of the author's proofs. Many text-books present a view of mathematical reasoning that is entirely erroneous. Here we are taught the true spirit of modern rigor, and the student who studies this book properly should know what true mathematics is. This is far more important than mere verbal accuracy of detail.

These pages are right from the lecture-room; not always in smooth clear polished style, but full of life and an enthusiasm that carries us along as we read them. They are well adapted to the classes in our American colleges, and we hope that they will be extensively used.

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## SHORTER NOTICES.

*Analytic Geometry of Space.* By VIRGIL SNYDER AND C. H. SISAM. New York, Henry Holt and Company, 1914. xi+289 pp.

THIS is one of the series of mathematical texts prepared under the general editorship of E. J. Townsend. The announced plan, however, of selecting as joint authors a mathematician and an engineer or physicist has not been adhered to in this case. As would be expected accordingly, the book will make its first and strongest appeal to the student of pure mathematics. If there is a single "practical problem" in the entire volume the reviewer has failed to discover it.

The authors are well fitted for their task since each is a specialist in the geometry of space and both are teachers of wide experience. Moreover their book possesses remarkable homogeneity of style and spirit, due possibly to the fact that the junior author was a pupil of the senior. At any rate, if there was any sharp division of labor the internal evidence is difficult to detect.

The book in size is an unpretentious volume of some 250 pages exclusive of the last chapter, and the modest preface states that it is intended as an introductory course. But even a casual examination will disclose an astonishing number and variety of topics, while a detailed reading emphatically confirms the first impression. Thus besides the usual equations of lines and planes and the metric formulas for angles and distances are introduced polar, cylindrical, and spherical coordinates, linear systems of planes, the notion of duality, homogeneous coordinates, and the plane at infinity, all in the first 37 pages! Surely this is information and ideas in a form sufficiently concentrated to stagger the average American undergraduate. It is only the very large number of excellent exercises (about 150 in the two chapters) which saves him from the otherwise inevitable confusion.