On any surface the two families of asymptotic curves are projectively equivalent, each lies on a quadric and is identically self-dual; the directrix curves are plane curves and one of the two families consists of conics. A non-degenerate quadric and a straight line, which is not a ruling of the quadric, constitute the focal surface of the directrix congruence of the first kind. The finite equations of the various associated loci are obtained.

ARNOLD DRESDEN,
Secretary of the Section.

NOTE ON FUNCTIONS OF SEVERAL COMPLEX VARIABLES.

BY PROFESSOR WILLIAM F. OSGOOD.

(Read before the American Mathematical Society, April 29, 1916.)

THE object of the present note is at once to extend the scope of a fundamental theorem of the theory of analytic functions of several complex variables and to simplify its proof.*

Definition.—Let S be the cylindrical region $(S_1, \cdots, S_n)$,

$$S_k: \qquad |z_k| < r_k \qquad (k = 1, \cdots, n);$$

let Σ be the region $(\Sigma_1, \cdots, \Sigma_n)$,

$$\Sigma_j: \qquad |z_j| < h_j < r_j \qquad (j = 1, 2);$$

$$\Sigma_k: \qquad |z_k| < r_k \qquad (k = 3, \cdots, n);$$

and let T be the region whose points are interior to S, but exterior to Σ:

$$T = S - \Sigma.\dagger$$

THEOREM. *Let $f(z_1, \cdots, z_n)$ be analytic throughout the region T. Then $f(z_1, \cdots, z_n)$ admits analytic continuation throughout S.*

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* The theorem was given by Kistler, "Ueber Funktionen von mehreren komplexen Veränderlichen," § 7, Basel, 1905, for the case that the excepted points lie on a finite number of analytic manifolds, each of $n - 2$ complex dimensions, and was proven by means of n-fold integrals.

† This symbolic form is suggestive, but not quite accurate, since it would assign to T certain of its boundary points, and T consists only of interior points.

The proof is given at once by Cauchy's integral formula for functions of a single variable—for simplicity we set $n = 3$—

$$f(z_1, z_2, z_3) = \frac{1}{2\pi i}\int_C \frac{f(z_1, t_2, z_3)dt_2}{t_2 - z_2}.$$

Here C shall be a circle,

$$|t_2| = r_2', \quad h_2 < r_2' < r_2,$$

r_2' being taken as near to r_2 as one pleases. Furthermore, z_1 shall be a point of the ring

$$h_1 < |z_1| < r_1,$$

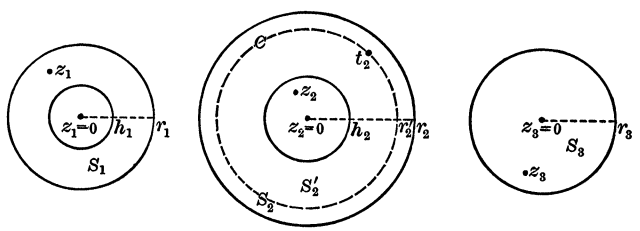

FIG. 1.

while z_3 is any point of the circle

$$|z_3| < r_3.$$

If finally z_2 is any point interior to the circle C, the hypotheses of the theorem justify the above formula.

But the integrand, for any fixed point t_2 on the circle C, is analytic throughout the whole region $S' = (S_1, S_2', S_3)$,

$$S_2': \qquad |z_2| < r_2';$$

and it is continuous when (z) lies in S' and t_2 on C. Hence the integral represents a function analytic throughout S'.

Thus $f(z_1, z_2, z_3)$ admits analytic continuation throughout S', and hence finally throughout S.

Remark. The foregoing theorem is contained in a theorem of Hartogs's.* Let a_1 be a point of the region $h_1 < |z_1| < r_1$, and let $a_3 = 0$. Then

(i) $f(z_1, z_2, z_3)$ is analytic in each point (a_1, z_2, a_3), where z_2 is any point of S_2', including the boundary;

* *Sitzungsber. der Münchener Akad.*, 36 (1906), p. 223. Cf. also the Madison Colloquium, pp. 168, 169.

(ii) $f(z_1, z_2, z_3)$ is analytic in each point (z_1, t_2, z_3), where t_2 is any point of C, and z_1, z_3 lie respectively in S_1 and S_3.

Hence $f(z_1, z_2, z_3)$ admits analytic continuation throughout S', and thus throughout S.

Hartogs's proof of the more general theorem is less simple, involving as it does n-fold integrals.

HARVARD UNIVERSITY,
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QUASI-PERIODICITY OF ASYMPTOTIC PLANE NETS.

BY DR. ALFRED L. NELSON.

(Read before the American Mathematical Society, April 21, 1916.)

1. *Introduction.*—The projective properties of plane nets of curves have been discussed by Wilczynski.* For this purpose he makes use of a certain completely integrable system of three linear homogeneous partial differential equations of the second order, namely,

$$
\begin{aligned}
y_{uu} &= ay_u + by_v + cy, \\
y_{uv} &= a'y_u + b'y_v + c'y, \qquad (1) \\
y_{vv} &= a''y_u + b''y_v + c''y.
\end{aligned}
$$

Three linearly independent solutions of this system, $y^{(k)}$ $(k = 1, 2, 3)$, are interpreted as the homogeneous coordinates of a point P_y which generates the plane net. The projective properties of the net are expressed in terms of the invariants of (1) under the transformations

$$
y = \lambda(u, v)\bar{y}; \qquad \bar{u} = U(u), \qquad \bar{v} = V(v). \tag{2}
$$

Two of these invariants,

$$
H = c' + a'b' - a_u', \quad K = c' + a'b' - b_v',
$$

the so-called Laplace-Darboux invariants, are expressed entirely in terms of the middle equation, which is of the type

* Wilczynski, "One-parameter families and nets of plane curves," *Transactions Amer. Math. Society*, vol. 12 (1911), no. 4, pp. 473–510.