(ii) $f(z_1, z_2, z_3)$ is analytic in each point (z_1, t_2, z_3), where t_2 is any point of C, and z_1, z_3 lie respectively in S_1 and S_3.

Hence $f(z_1, z_2, z_3)$ admits analytic continuation throughout S', and thus throughout S.

Hartogs's proof of the more general theorem is less simple, involving as it does n-fold integrals.

HARVARD UNIVERSITY,
April 17, 1916.

QUASI-PERIODICITY OF ASYMPTOTIC PLANE NETS.

BY DR. ALFRED L. NELSON.

(Read before the American Mathematical Society, April 21, 1916.)

1. *Introduction.*—The projective properties of plane nets of curves have been discussed by Wilczynski.* For this purpose he makes use of a certain completely integrable system of three linear homogeneous partial differential equations of the second order, namely,

$$
\begin{aligned}
y_{uu} &= ay_u + by_v + cy,\\
y_{uv} &= a'y_u + b'y_v + c'y, \qquad (1)\\
y_{vv} &= a''y_u + b''y_v + c''y.
\end{aligned}
$$

Three linearly independent solutions of this system, $y^{(k)}$ $(k = 1, 2, 3)$, are interpreted as the homogeneous coordinates of a point P_y which generates the plane net. The projective properties of the net are expressed in terms of the invariants of (1) under the transformations

$$(2) \qquad y = \lambda(u, v)\bar{y}; \qquad \bar{u} = U(u), \qquad \bar{v} = V(v).$$

Two of these invariants,

$$H = c' + a'b' - a_u', \quad K = c' + a'b' - b_v',$$

the so-called Laplace-Darboux invariants, are expressed entirely in terms of the middle equation, which is of the type

* Wilczynski, "One-parameter families and nets of plane curves," *Transactions Amer. Math. Society*, vol. 12 (1911), no. 4, pp. 473–510.

studied by Laplace.* A certain special case, namely, when the invariants H and K are equal, acquires especial interest in view of a theorem of Koenigs:† *The perspectives of the asymptotic curves of a surface from a fixed point on a fixed plane form a net with equal invariants. Conversely, a plane net with equal invariants may be regarded as the perspectives from a fixed point of the asymptotic curves of a surface.* We shall speak of a plane net for which $H = K$ as an *asymptotic* plane net. The present paper has as its object the discussion of the Laplace transformation, to which a large part of Wilczynski's paper is devoted, for the special case mentioned. The characteristic system of partial differential equations for the general Laplace transformed net will be computed, together with its fundamental invariants, and certain theorems concerning quasi-periodic plane nets will be deduced.

The first Laplace transform is the net generated by the covariant point $P_{y'}$, whose coordinates are $y'^{(k)} = y_v^{(k)} - a'y^{(k)}$ $(k = 1, 2, 3)$. The minus first Laplace transform is described by the covariant point $P_{y^{(-1)}}$, where $y^{(-1)} = y_u - b'y$. Each of these new nets has a first and a minus first Laplace transform, and it is readily shown that the minus first transform of the first transform, as well as the first transform of the minus first transform, is the original net. Accordingly, we have, in general, an infinite chain of nets, the Laplace suite, covariantly connected with the original net. Each net of the suite has, of course, a characteristic system of equations, and we shall call the system of the ith transform $(Y^{(i)})$ (i a positive or negative integer), and distinguish the coefficients and invariants of this transform by the subscript i.

Under the assumption of the equality of H and K, the system (1) may be put in the unique form

$$
\begin{aligned}
& y_{uu} = -2\frac{\delta_u}{\delta}\cdot y_u + \mathfrak{B}y_v - \mathfrak{B}\frac{\partial}{\partial v}\log\frac{\mathfrak{B}}{\delta^2}\cdot y, \\
(3) \qquad (Y) \quad & y_{uv} = Hy, \quad \left(H = \mathfrak{A}''\mathfrak{B} - \frac{\partial^2 \log \delta^2}{\partial u \partial v}\right), \\
& y_{vv} = \mathfrak{A}''y_u - 2\frac{\delta_v}{\delta}\cdot y_v - \mathfrak{A}''\frac{\partial}{\partial u}\log\frac{\mathfrak{A}''}{\delta^2}\cdot y,
\end{aligned}
$$

* "Recherches sur le calcul intégral aux différences partielles." Oeuvres de Laplace, t. IX, pp. 29, et seq.

† Koenigs, "Sur les réseaux plans à invariants égaux et les lignes asymptotiques." *Comptes Rendus*, vol. 114 (1892), p. 55.

which is characterized by the relations $a' = b' = 0$, and where δ is a non-vanishing function of u and v.* The integrability conditions of (Y) are obtained from the identical relations

$$(y_{uu})_v = (y_{uv})_u; \quad (y_{uv})_v = (y_{vv})_u.$$

They are given, for the general case, as equations (5) of Wilczynski's paper. Each of the other nets of the Laplace suite has the six corresponding integrability conditions, which are satisfied as a result of those of the original net. We shall, however, use the integrability conditions of the other nets of the suite as being, by their form, better adapted to the simplification of our computations, and shall refer to them as the first, second, etc., the order being understood to be the same as in the case of those of the original net, as given by Wilczynski.

2. *The General Laplace Transform.*—Let us assume that the $(k+1)$th (k a positive integer) transform is non-degenerate,† and has the following coefficients:

$$
\begin{aligned}
&a_{k+1} = \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_k''! \, H_k!}{\delta^2}, \qquad b_{k+1} = \frac{H_k}{\mathfrak{A}_k''},\\
&a_{k+1}' = \frac{\partial}{\partial v} \log H_k!, \qquad b_{k+1}' = 0,\\
&a_{k+1}'' = \mathfrak{A}_{k+1}'', \qquad b_{k+1}'' = \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{\delta^2},\\
(4) \qquad &c_{k+1} = -\frac{H_k}{\mathfrak{A}_k''} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_{k-1}''!}{H_{k-1}! \, \delta^2},\\
&c_{k+1}' = H_k,\\
&c_{k+1}'' = c_k'' + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}_{k-1}''!}{(H_{k-1}!)^2 \delta^2}\\
&\qquad\qquad - \frac{\partial}{\partial v} \log \mathfrak{A}_k'' \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_{k-1}''!}{H_{k-1}! \, \delta^2},
\end{aligned}
$$

* The transformation which accomplishes this, with the help of the integrability conditions, is of the form of the first of (2), and does not alter the net.

† It is easily shown that for the $(k+1)$th transform to degenerate into a single curve it is necessary and sufficient that $\mathfrak{A}_k'' H_k = 0$ (cf. (5)), assuming that the kth transform is not degenerate. The assumption that the $(k+1)$th transform is non-degenerate carries with it, of course, the assumption of the non-degeneracy of all the transforms up to the $(k+1)$th.

where

$$(5)\qquad \begin{aligned} H_j &= H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log H_{j-1}!, \\ \mathfrak{A}_j'' &= \frac{\mathfrak{A}_{j-1}''}{H_{j-1}} \left(H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{A}_{j-1}''!}{\delta^2} \right), \end{aligned}$$

and

$$\theta_j! = \theta_0 \theta_1 \theta_2 \cdots \theta_j, \quad \theta_0 = \theta,$$
$$(j \text{ a positive integer}, \ \theta = \mathfrak{A}'' \text{ or } H).$$

The coefficients of the first and second transforms, if these transforms are not degenerate, can readily be shown to be of the form (4).

The $(k+2)$th transform is generated by the point $P_{y^{(k+2)}}$, where

$$(6)\qquad y^{(k+2)} = y_v^{(k+1)} - \frac{\partial}{\partial v} \log H_k! \cdot y^{(k+1)}.$$

By differentiation of (6) and application of the equations given by (4), we find the relations

$$(7)\qquad \begin{aligned} y_u^{(k+2)} &= H_{k+1} y^{(k+1)}, \\ y_v^{(k+2)} &= \mathfrak{A}_{k+1}'' y_u^{(k+1)} + \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\, \delta^2} \cdot y_v^{(k+1)} \\ &\qquad + \left(c_{k+1}'' - \frac{\partial^2}{\partial v^2} \log H_k! \right) y^{(k+1)}. \end{aligned}$$

Except when $\mathfrak{A}_{k+1}'' H_{k+1} = 0$, that is, except when the $(k+2)$th transform is degenerate, we may solve equations (6) and (7) for $y^{(k+1)}$, $y_u^{(k+1)}$, $y_v^{(k+1)}$, obtaining the expressions

$$(8)\qquad \begin{aligned} y^{(k+1)} &= \frac{1}{H_{k+1}} y_u^{(k+2)}, \\ y_v^{(k+1)} &= y^{(k+2)} + \frac{1}{H_{k+1}} \frac{\partial}{\partial v} \log H_k! \cdot y_u^{(k+2)}, \\ y_u^{(k+1)} &= \frac{1}{H_{k+1}} \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_{k+1}''!\, H_k!}{\delta^2} \cdot y_u^{(k+2)} \\ &\qquad + \frac{1}{\mathfrak{A}_{k+1}''} y_v^{(k+2)} - \frac{1}{\mathfrak{A}_{k+1}''} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\, \delta^2} \cdot y^{(k+2)}, \end{aligned}$$

where the fourth integrability condition of $(Y^{(k+1)})$ has been used to obtain the given form of the coefficient of $y_u^{(k+2)}$ in the third equation of (8).

If we differentiate equations (7) again, and make use of (4), we find the following equations:

$$y_{uu}^{(k+2)} = (H_{k+1})_u \cdot y^{(k+1)} + H_{k+1} \cdot y_u^{(k+1)},$$

$$y_{uv}^{(k+2)} = (H_{k+1})_v \cdot y^{(k+1)} + H_{k+1} \cdot y_v^{(k+1)},$$

(9)
$$\begin{aligned} y_{vv}^{(k+2)} = &\Big(c''_{k+1} + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}_k''!}{(H_k!)^2\delta^2} \\ &+ \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{\delta^2} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2} \Big) y_v^{(k+1)} \\ &+ \mathfrak{A}''_{k+1} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k+1}!}{\delta^2} \cdot y_u^{(k+1)} \\ &+ \Big(c''_{k+1} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!\, c''_{k+1}}{H_k!\,\delta^2} - \frac{\partial^3}{\partial v^3} \log H_k! \\ &+ H_k \mathfrak{A}''_{k+1} \Big) y^{(k+1)}. \end{aligned}$$

Substitution of (8) in (9) yields the characteristic system of equations of the $(k + 2)$th transform, for which the following are the coefficients:

(10)
$$\begin{aligned} a_{k+2} &= \frac{\partial}{\partial u} \log \frac{\mathfrak{A}''_{k+1}!\, H_{k+1}!}{\delta^2}, & b_{k+2} &= \frac{H_{k+1}}{\mathfrak{A}''_{k+1}}, \\ a'_{k+2} &= \frac{\partial}{\partial v} \log H_{k+1}!, & b'_{k+2} &= 0, \\ a''_{k+2} &= \mathfrak{A}''_{k+2}, & b''_{k+2} &= \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k+1}!}{\delta^2}, \\ c_{k+2} &= -\frac{H_{k+1}}{\mathfrak{A}''_{k+1}} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2}, \\ c'_{k+2} &= H_{k+1}, \\ c''_{k+2} &= c''_{k+1} + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}_k''!}{(H_k!)^2\delta^2} \\ &\qquad - \frac{\partial}{\partial v} \log \mathfrak{A}''_{k+1} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2}, \end{aligned}$$

where the expression for a''_{k+2} (cf. (5)) is given by the fifth integrability condition of $(Y^{(k+1)})$. Comparison of (10) with

(4) shows that we have completed the induction proof that (4) gives the characteristic system $(Y^{(k+1)})$ for the $(k+1)$th transform (k positive).

We may compute the general minus $(k+1)$th transform in the same manner, or may more easily obtain the result by applying the substitution

$$\begin{pmatrix} \mathfrak{A}_j'' & u & c_k'' \\ \mathfrak{B}_{-j} & v & c_k \end{pmatrix} \quad (j = k-1, k, k+1)$$

to the equations given by (4). The coefficients of the minus $(k+1)$th (k positive) turn out to be

$$a_{-(k+1)} = \frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-k}!}{\delta^2}, \quad b_{-(k+1)} = \mathfrak{B}_{-(k+1)},$$

$$c_{-(k+1)} = c_{-k} + \frac{\partial^2}{\partial u^2}\log\frac{\mathfrak{B}_{-(k-1)}!}{(H_{k-1}!)^2\delta^2} - \frac{\partial}{\partial u}\log\mathfrak{B}_{-k}\cdot\frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-(k-1)}!}{H_{k-1}!\,\delta^2},$$

$$(11) \qquad a'_{-(k+1)} = 0, \qquad b'_{-(k+1)} = \frac{\partial}{\partial u}\log H_k!,$$

$$c'_{-(k+1)} = H_k,$$

$$a''_{-(k+1)} = \frac{H_k}{\mathfrak{B}_{-k}}, \quad b''_{-(k+1)} = \frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-k}!\,H_k!}{\delta^2},$$

$$c''_{-(k+1)} = -\frac{H_k\partial}{\mathfrak{B}_{-k}\partial u}\log\frac{\mathfrak{B}_{-(k-1)}!}{H_{k-1}!\,\delta^2},$$

where

$$(12) \qquad \mathfrak{B}_{-j} = \frac{\mathfrak{B}_{-(j-1)}}{H_{j-1}}\left(H_{j-1} - \frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{B}_{-(j-1)}!}{\delta^2}\right)$$

(j any positive integer),

and

$$\mathfrak{B}_{-j}! = \mathfrak{B}_0\mathfrak{B}_{-1}\mathfrak{B}_{-2}\cdots\mathfrak{B}_{-j}.$$

We have already mentioned the fact that for the $(k+2)$th transform to degenerate, while the $(k+1)$th transform is non-degenerate, it is necessary and sufficient that $\mathfrak{A}_{k+1} = 0$, or $H_{k+1} = 0$. Reference to the third equation of the system determined by (4) shows that *the first of these conditions is* *necessary and sufficient for the curves* $u = \text{const.}$ *of the* $(k+1)$*th transform to be straight lines.* The first equation of (4) shows that *the curves* $v = \text{const.}$ *of the* $(k+1)$*th transform cannot be straight lines,* since the condition for this, $b_{k+1} = 0$, would make the $(k+1)$th transform degenerate. Similarly, *degeneracy of the minus* $(k+2)$*th transform is equivalent to* $\mathfrak{B}_{-(k+1)} = 0$, *or* $H_{k+1} = 0$. *The first of these conditions is necessary and sufficient for the curves* $v = \text{const.}$ *of the minus* $(k+1)$*th transform to be straight lines. The curves* $u = \text{const.}$ *of this transform cannot be straight lines.*

3. *Summary of Fundamental Invariants.*—In the paper referred to, Wilczynski has proved the theorem: *If the invariants* $\mathfrak{B} = b$, $\mathfrak{C} = c + a'b + ab' - b'^2 - b_u'$, $\mathfrak{A}' = \frac{2}{3}a' - \frac{1}{3}b'' + \frac{1}{6}(a_v''/a'')$, $\mathfrak{B}' = \frac{2}{3}b' - \frac{1}{3}a + \frac{1}{6}(b_u/b)$, $\mathfrak{C}' = c' + a'b' - \frac{1}{3}(a_v + b_v')$, $\mathfrak{A}'' = a''$, $\mathfrak{C}'' = c'' + a''b' + a'b'' - a'^2 - a_v'$, *of a net are given as functions of* u *and* v, *subject to the integrability conditions, the net is determined, except for a projective transformation.** Omitting the details of computation, these fundamental invariants, together with the invariants H and K, for the $(k+1)$th transform have the following expressions:

$$
\begin{aligned}
\mathfrak{B}_{k+1} &= \frac{H_k}{\mathfrak{A}_k''}, \\
\mathfrak{C}_{k+1} &= -\frac{H_k}{\mathfrak{A}_k''} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_{k-1}''!}{(H_{k-1}!)^2 H_k \delta^2}, \\
\mathfrak{A}_{k+1}' &= \frac{1}{6}\frac{\partial}{\partial v} \log \frac{(H_k!)^4 \delta^4 \mathfrak{A}_{k+1}''}{(\mathfrak{A}_k''!)^2}, \\
\mathfrak{B}_{k+1}' &= \frac{1}{6}\frac{\partial}{\partial u} \log \frac{H_k \delta^4}{(\mathfrak{A}_k''!)^2 \mathfrak{A}_k'' (H_k!)^2}, \qquad (13) \\
\mathfrak{C}_{k+1}' &= H_k - \frac{1}{3}\frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{A}_k''!\, H_k!}{\delta^2}, \\
\mathfrak{A}_{k+1}'' &= \mathfrak{A}_{k+1}'', \\
\mathfrak{C}_{k+1}'' &= -\mathfrak{A}_{k+1}'' \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_{k+1}''!\, H_k!}{\delta^2}, \\
H_{k+1} &= H_{k+1}, \qquad K_{k+1} = H_k.
\end{aligned}
$$

* P. 485 of Wilczynski's paper.

The corresponding invariants of the minus $(k+1)$th transform are

$$
(14)\qquad
\begin{aligned}
&\mathfrak{B}_{-(k+1)} = \mathfrak{B}_{-(k+1)},\\
&\mathfrak{C}_{-(k+1)} = -\,\mathfrak{B}_{-(k+1)}\frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-(k+1)}!\,H_k!}{\delta^2},\\
&\mathfrak{A}'_{-(k+1)} = \frac{1}{6}\frac{\partial}{\partial v}\log\frac{H_k\delta^4}{(\mathfrak{B}_{-k}!)^2\mathfrak{B}_{-k}(H_k!)^2},\\
&\mathfrak{B}'_{-(k+1)} = \frac{1}{6}\frac{\partial}{\partial u}\log\frac{(H_k!)^4\mathfrak{B}_{-(k+1)}\delta^4}{(\mathfrak{B}_{-k}!)^2},\\
&\mathfrak{C}'_{-(k+1)} = H_k - \frac{1}{3}\frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{B}_{-k}!\,H_k!}{\delta^2},\\
&\mathfrak{A}''_{-(k+1)} = \frac{H_k}{\mathfrak{B}_{-k}},\\
&\mathfrak{C}''_{-(k+1)} = -\frac{H_k}{\mathfrak{B}_{-k}}\cdot\frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-(k-1)}!\,H_k}{(H_k!)^2\delta^2},\\
&H_{-(k+1)} = H_k, \qquad K_{-(k+1)} = H_{k+1}.
\end{aligned}
$$

4. *Quasi-periodic Nets.*—In case the ith transform is projectively equivalent to the $(i+j)$th transform (j a positive integer, i a positive or negative integer), the net is said to be quasi-periodic, of period j. Let us assume that $\mathfrak{A}_k''$, $\mathfrak{B}_{-k}$ and H_k are different from zero, so that the $(k+1)$th and minus $(k+1)$th transforms are non-degenerate. A necessary and sufficient condition for a quasi-periodic net, of period $2(k+1)$, is that the fundamental invariants of $(Y^{(k+1)})$ be equal to the corresponding invariants of $(Y^{-(k+1)})$. If such a period exists, however, the invariants H of the two transforms must also be equal. By reference to (5), we see that this condition is

$$
(15)\qquad \frac{\partial^2}{\partial u\partial v}\log H_k! = 0,
$$

so that by a transformation of the form $\bar{u} = U(u)$, $\bar{v} = V(v)$ we may make $H_k! = 1$. But (15) also makes the $(k+1)$th and minus $(k+1)$th transforms asymptotic, so that we may take advantage of the following theorem: *If the coefficients a, b, a'', b'', of the form* (3) *of the differential equations of an* *asymptotic plane net are given as functions of u and v, subject to the integrability conditions, they determine the net except for a projective transformation.** For the case of quasi-period $2(k+1)$, the transformation which makes $H_k! = 1$ also puts the differential equations of the $(k+1)$th and the minus $(k+1)$th transforms, namely, those given by (10) and (11), in the form (3). Hence we obtain the further conditions desired by equating the coefficients a_{k+1}, b_{k+1}, a''_{k+1}, b''_{k+1}, to the corresponding coefficients of the minus $(k+1)$th transform. Remembering that $H_k! = 1$, these conditions are the following:

$$
(16)\qquad
\begin{gathered}
\frac{\partial}{\partial u}\log\frac{\mathfrak{A}_k''!}{\delta^2} = \frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-k}!}{\delta^2},\\
\frac{H_k}{\mathfrak{A}_k''} = \mathfrak{B}_{-(k+1)},\\
\mathfrak{A}''_{k+1} = \frac{H_k}{\mathfrak{B}_{-k}},\\
\frac{\partial}{\partial v}\log\frac{\mathfrak{A}_k''!}{\delta^2} = \frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-k}!}{\delta^2}.
\end{gathered}
$$

The first and fourth of these equations imply $\mathfrak{A}_k''!/\mathfrak{B}_{-k}! =$ constant, and we may make this constant equal to unity by a suitable transformation of the independent variables, without violating the condition $H_k! = 1$. The second and third equations of (16) are now equivalent, in view of (5) and (12), so that we have the theorem: *If an asymptotic plane net, for which the $(k+1)$th and minus $(k+1)$th transforms are not degenerate, is quasi-periodic, of period $2(k+1)$, its differential equations may be so written that*

$$
(17)\qquad H_k! = 1,\quad \mathfrak{A}_k''! = \mathfrak{B}_{-k}!,\quad \mathfrak{A}_{k+1}'' = H_k/\mathfrak{B}_{-k}.
$$

Conversely, any asymptotic plane net, whose $(k+1)$th and minus $(k+1)$th transforms are non-degenerate, and for which equations (17) *hold, is quasi-periodic, of period* $2(k+1)$.

In order to discuss the case of odd quasi-periods, we equate the corresponding fundamental invariants of $(Y^{(-k)})$ and $(Y^{(k+1)})$, to obtain the conditions for a quasi-period $2k+1$. The conditions are the following:

* The truth of this theorem is easily seen by reference to (3).

$$\mathfrak{B}_{-k} = \frac{H_k}{\mathfrak{A}_k''},$$

$$-\mathfrak{B}_{-k}\frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-k}!\,H_{k-1}!}{\delta^2} = -\frac{H_k}{\mathfrak{A}_k''}\cdot\frac{\partial}{\partial v}\log\frac{\mathfrak{A}_{k-1}''!\,H_k}{(H_k!)^2\delta^2},$$

$$\frac{1}{6}\frac{\partial}{\partial v}\log\frac{\delta^4 H_{k-1}}{(\mathfrak{B}_{-(k-1)}!)^2\mathfrak{B}_{-(k-1)}(H_{k-1}!)^2} = \frac{1}{6}\frac{\partial}{\partial v}\log\frac{(H_k!)^4\delta^4\mathfrak{A}_{k+1}''}{(\mathfrak{A}_k''!)^2},$$

$$\text{(18)}\quad \frac{1}{6}\frac{\partial}{\partial u}\log\frac{(H_{k-1}!)^4\delta^4\mathfrak{B}_{-k}}{(\mathfrak{B}_{-(k-1)}!)^2} = \frac{1}{6}\frac{\partial}{\partial u}\log\frac{\delta^4 H_k}{(\mathfrak{A}_k''!)^2\mathfrak{A}_k''(H_k!)^2},$$

$$H_{k-1} - \frac{1}{3}\frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{B}_{-(k-1)}!\,H_{k-1}!}{\delta^2} = H_k - \frac{1}{3}\frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{A}_k''!\,H_k!}{\delta^2},$$

$$\frac{H_{k-1}}{\mathfrak{B}_{-(k-1)}} = \mathfrak{A}_{k+1}'',$$

$$-\frac{H_{k-1}}{\mathfrak{B}_{-(k-1)}}\cdot\frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-(k-2)}!\,H_{k-1}}{(H_{k-1}!)^2\delta^2} = -\mathfrak{A}_{k+1}''\frac{\partial}{\partial u}\log\frac{\mathfrak{A}_{k+1}''!\,H_k!}{\delta^2}.$$

Since also $H_{-k} = H_{k+1}$, we have, in view of (5),

$$\text{(19)}\qquad \frac{\partial^2}{\partial u\partial v}\log H_{k-1}!\,H_k! = 0,$$

so that by a suitable transformation of the independent variables we may make

$$\text{(20)}\qquad H_{k-1}!\,H_k! = 1.$$

By use of (5) and (12), we find that substitution of (20) and the first of (18) in the remaining equations causes (18) to yield only one new condition, namely, $\mathfrak{A}_k''!\,H_{k-1}!/\mathfrak{B}_{-(k-1)}! =$ constant. We may make this constant equal to unity by a transformation of the independent variables, without violating the condition (20). Hence the following theorem results. *If an asymptotic plane net whose* $(k+1)$*th and minus* $(k+1)$*th* *transforms are not degenerate, is quasi-periodic, of period* $2k+1$, *its differential equations may be so written that*

$$H_{k-1}!\,H_k! = 1, \quad \mathfrak{A}_k''\mathfrak{B}_{-k} = H_k, \tag{21}$$
$$\mathfrak{A}_k''!\,H_{k-1}! = \mathfrak{B}_{-(k-1)}.$$

Conversely, any asymptotic plane net, whose $(k+1)$*th and minus* $(k+1)$*th transforms are non-degenerate, and for which equations* (21) *hold, is quasi-periodic, of period* $2k+1$.

ANN ARBOR, MICH.,
April, 1916.

CONCERNING HILL'S DERIVATION OF THE LAGRANGE EQUATIONS OF MOTION.

BY PROFESSOR K. P. WILLIAMS.

(Read before the American Mathematical Society, April 22, 1916.)

THERE are two methods of deriving the Lagrange equations of motion that are commonly given in treatises on dynamics. One of the methods makes use of what is known as Hamilton's principle, while the other proceeds directly from D'Alembert's equation by means of a transformation of variables. While the first method leaves little to be desired as regards elegance, it makes use of a principle not essential to an understanding of the equations of motion or of their application. The second method, as usually given, involves a considerable amount of calculation.

In a paper entitled "On the differential equations of dynamics," in the first volume of the *Analyst*,* Hill sought to derive the Lagrange equations from D'Alembert's equation without making use of the details of the calculation above mentioned. For some reason his ideas do not seem to have found their way into the literature of the subject. In the form in which he presented it, Hill's derivation seems to me to be open to criticism on account of some of the assumptions that he makes. It is possible, however, to avoid making these assumptions, and when this is done a very simple and direct derivation of the Lagrange equations is obtained.

We start with D'Alembert's equation

* Collected Papers, vol. 1, pp. 192–194.