Here $e_2 \oplus e_3 \neq e_3 \oplus e_2$.

$\overline{\text{III}}_b$.

$\oplus$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_2	e_2
e_3	e_3	e_2	e_3

$\odot$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_1	e_2	e_2
e_3	e_1	e_3	e_3

Here $e_2 \odot e_3 \neq e_3 \odot e_2$.

UNIVERSITY OF CALIFORNIA,
March, 1916.

NOTE ON REGULAR TRANSFORMATIONS.

BY DR. L. L. SILVERMAN.

LET $u(x)$ be bounded and integrable, $0 \leqq x$, and $k(x, y)$ integrable in y for each x, $0 < y \leqq x$; then the transformation*

$$v(x) = \alpha u(x) + \int_0^x k(x, s)u(s)ds \tag{1}$$

is regular if

$$\lim_{x=\infty} u(x)$$

implies the existence of

$$\lim_{x=\infty} v(x)$$

and the equality of the limits. The transformation (1), which depends on the number α and on the function $k(x, y)$, will be denoted by the symbol $[\alpha; k(x, y)]$. Examples of regular transformations are given by [1; 0], which is the identical transformation, and $[0; 1/x]$, which corresponds to the first Hölder mean. In a forthcoming paper† the author discusses conditions on α and $k(x, y)$ for the regularity of the transformation‡ (1), and proves the following theorem:||

THEOREM 1. *A sufficient condition that* $k(x, y)$ *defined,* $0 < y \leqq x$, *and integrable in* y *for each* x, *correspond to a*

* It is assumed that the improper integral converges; the lower limit of integration is taken zero for convenience.

† *Transactions*, vol. 17 (1916).

‡ The function $k(x, y)$ in (1) is $(1 - \alpha)$ times the function $k(x, y)$ in the article referred to.

|| See Theorem III in the article referred to; the numbers a and b of that theorem are here replaced by 0 and a respectively. The right-hand member of the last condition is $1 - \alpha$ instead of unity; see preceding footnote.

regular transformation is

$$\int_0^a |k(x, y)|dy \text{ converges}, \quad \lim_{x=\infty} \int_0^a |k(x, y)|dy = 0,$$

$$\int_0^x |k(x, y)|dy < A, \quad x > 0, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where a and A are positive constants.

We wish in this note to consider some special cases of Theorem 1.

THEOREM 2. *Let $k(x, y)$ be defined, $0 < y \leqq x$, and integrable in y for each x; then a sufficient condition* that the transformation $[\alpha; k(x, y)]$ be regular is that*

$$|k(x, y)| \leqq \frac{Mf'(y)}{f(x)}, \quad 0 < y \leqq x, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where $f(x)$ is a function continuous, $x \geqq 0$, and having a continuous derivative, $x > 0$, and satisfying the conditions

$$f(x) \geqq 0, \quad x \geqq 0; \quad f'(x) \geqq 0, \quad x > 0; \quad \lim_{x=\infty} f(x) = \infty;$$

and where

$$M \geqq 0.$$

The fourth condition of Theorem 1 is satisfied by hypothesis. The first condition is satisfied since from the hypothesis it follows that the integral of $k(x, y)$ converges absolutely. Furthermore,

$$\int_0^a |k(x, y)|dy \leqq \frac{Mf(a)}{f(x)}, \quad x > 0.$$

The second and third conditions of Theorem 1 follow at once from this inequality. Thus all the conditions of Theorem 1 are satisfied.

COROLLARY 1. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a regular transformation is*

$$|k(x, y)| \leqq \frac{M}{(1 + y) \log (1 + x)}, \quad 0 < y \leqq x,$$

$$\lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where $M \geqq 0$.

* The convergence of the integral in the second condition—in fact, its absolute convergence—follows from the first condition.

Taking $f(x) = \log (1 + x)$, it is seen that the conditions of Theorem 2 are satisfied.

COROLLARY 2. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a regular transformation is*

$$|k(x, y)| \leqq \frac{M}{x^{1-p}y^p}, \quad 0 < y \leqq x, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where p and M are constants, $0 \leqq p < 1$, $M \geqq 0$.

Taking $f(x) = x^{1-p}$, it is seen that the conditions of Theorem 2 are satisfied.

Similar theorems hold for transformations of sequences.

CORNELL UNIVERSITY.

SHORTER NOTICES.

Contributions to the Founding of the Theory of Transfinite Numbers. By GEORG CANTOR. Translated, and provided with an introduction and notes, by PHILIP E. B. JOURDAIN. Chicago and London, Open Court Publishing Company, 1915. ix+211 pp.

THIS volume contains a translation of G. Cantor's two fundamental memoirs on transfinite numbers which appeared in the *Mathematische Annalen* for 1895 and 1897 under the title "Beiträge zur Begründung der transfiniten Mengenlehre." The translator has changed the title to that given above because "these memoirs are chiefly occupied with the investigation of the various transfinite cardinal and ordinal numbers." The book is put forth by the publishers as number 1 of "The Open Court Series of Classics of Science and Philosophy."

It is not too much to say that the work of Cantor on the theory of classes of points has brought about both a mathematical and a philosophical revolution, that in philosophy perhaps being even greater than that in mathematics, notwithstanding the fact that "these theories of Cantor are permeating modern mathematics."*

In the opinion of the translator K. Weierstrass, R. Dedekind, and G. Cantor are the three men who have exerted the most marked influence on modern pure mathematics and indirectly on the modern logic and philosophy which abut on it. In

* E. H. Moore, Introduction to a Form of General Analysis, p. 2.