THE MADISON COLLOQUIUM LECTURES ON MATHEMATICS

Part II. Topics in the Theory of Functions of Several Complex Variables. By WILLIAM FOGG OSGOOD. New York, American Mathematical Society, 1914.

THE transition from the theory of functions of a single complex variable to the analogous theory for two variables corresponds in the domain of reals to a transition from functions on a plane to functions whose arguments are points of a four dimensional space. Generalizations from two to three real dimensions frequently involve great difficulties, even with the aid of our geometrical intuitions of the plane and three-space. It is not surprising therefore to find that familiar theorems concerning functions of a single complex variable either have no analogues at all, or else yield a multiplicity of generalizations, when the number of independent variables is increased. In the plane, for example, we have only the theory of connectivity associated with curves, while in space of four dimensions there are three kinds of connectivity corresponding in an analogous way to curves, surfaces, and three-spreads. On these subspaces of different dimensions there are theories of integrals corresponding to the curve integrals which play such a fundamental rôle in the theory of functions of a single complex variable.

The increase in complexity mentioned in the preceding paragraph arises from the greater multiplicity of geometrical configurations in the higher spaces. But there are also serious analytical obstacles to the extensions of the theory of functions of a single complex variable. Thus for a function $f(z_1, z_2)$ of two variables $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ the familiar potential equation is replaced by the system

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} = 0, \quad \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_2^2} = 0,$$

$$\frac{\partial^2 u}{\partial x_1 \partial y_2} - \frac{\partial^2 u}{\partial x_2 \partial y_1} = 0, \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial y_1 \partial y_2} = 0,$$

which admits of a much greater variety of initial conditions and is in other respects also more formidable.

A typical example of a theorem which breaks down entirely in the higher spaces is the following: Let $\varphi(x, y)$ be a function of two complex variables analytic at the origin and having $\varphi(0, y) \equiv 0$. With the help of the preparation theorem of Weierstrass φ can always be decomposed into a product of the form

$$\varphi(x, y) = P(x, y)M(x, y),$$

where $P(x, y)$ is a polynomial in y with coefficients which are series in x without constant terms, while M is a series in x and y with non-vanishing constant term. In a suitably chosen neighborhood of the origin, therefore, the function $y(x)$ defined by the equation $\varphi = 0$ is finitely multiple valued and has its values defined by the polynomial equation $P(x, y) = 0$. In the Colloquium Lectures Professor Osgood raises the question as to whether a similar theorem would be true for a function $y(x_1, x_2, \cdots, x_n)$ defined by an equation of the form

$$\varphi(x_1, x_2, \cdots, x_n, y) = 0. \tag{1}$$

He has more recently discovered that the theorem is not true when $n > 1$,* and the example by means of which he proves this statement provides some very interesting further information. An equation of the form (1) may in fact define a function $y(x_1, \cdots, x_n)$ which is single-valued and monogenic in a part of a neighborhood of the origin, while the origin itself is a point of a natural boundary of the function, other parts of the neighborhood being lacunary spaces. Such a function could clearly not be analytic or even algebroid at the origin.

The foregoing remarks will indicate to some extent the complexity of the questions which lie at the foundations of the theory of functions of several complex variables. Professor Osgood has been a thorough student of the subject with a keen appreciation of its deficiencies at the present stage of development. His Lectures seem to be a resumé preliminary to the publication of an exhaustive treatise, and it is much to be hoped that such a treatise may appear from his pen in the near future. Lecture I contains a historical introduction and "general survey" of the field, while the other four lectures are devoted to a descriptive and critical exposition of fundamental theorems, some of the most sig-

* *Transactions Amer. Math. Society*, vol. 17 (1916), p. 4.

nificant of which are the result of the author's own researches. The whole will be most helpful to students of the subject.

In the first lecture the author mentions a definition of a function of several complex variables published by Cauchy in 1831, but the earliest phases of the theory to be studied in detail were associated with special problems. Among the more important of these were the extensions of the existence theorems of Cauchy to partial differential equations and systems of equations defining implicit functions of several complex variables, the periodic abelian functions arising from Jacobi's problem of inversion, and periodic functions of several variables in general. The existence theorems did not require the development of new methods, their extensions being obtained by means of the famous device of dominating series applied by Cauchy to the simpler cases. The masterly generalization of the elliptic inversion problem by Jacobi was, however, the precursor of a long series of investigations during one of the most fruitful periods in the annals of mathematics. It was during this period that the theory of the theta functions of several variables, which lies at the heart of the periodic function theory, was so highly developed by Riemann and Weierstrass.

The results which have just been mentioned were the products of successful attempts to generalize the elliptic functions and elliptic theta functions. After the development of the theory of automorphic functions of a single variable by Klein and Poincaré in the early eighties, it was natural that similar generalizations should be sought for functions of several variables. Picard was the pioneer in this field. The functions studied when, for example, there are two independent complex variables x, y are those which are invariant under groups of projective transformations, or groups of transformations of the form

$$x' = \frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, \quad y' = \frac{\alpha_2 y + \beta_2}{\gamma_2 y + \delta_2}.$$

The theory of these functions in turn stimulated the interest of Picard and his associates in algebraic functions of two variables and the surface and line integrals of various kinds associated with them. Their results are analogous in a general way to those of the theory of algebraic functions of a single variable, although the theorems in particular cases are strikingly different.

Lecture II is devoted to a rather heterogeneous collection of theorems. It begins with an abbreviated account of a generalization of Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t - z}$$

and the theory of residues. Several sections are then devoted to a discussion of theorems giving sufficient conditions that a function of several complex variables be analytic, rational, or algebraic. Weierstrass stated the theorem that a function $f(z_1, z_2, \cdots, z_n)$ of several complex variables is surely rational if it is representable near every point of analysis by the quotient of two functions analytic and prime to each other at the point in question. He did not specify explicitly the character of the infinite region implied in the phrase "at every point of analysis," but he seems to have intended that each of the variables $z_1, z_2, \cdots, z_n$ should range independently over a plane extended by a single point at infinity. Professor Osgood has called attention to the fact that the part of the hypothesis of the theorem referring to the infinite region is under these circumstances equivalent to presupposing properties of the function f at finite points into which the points of the infinite region have been converted by a transformation of the form

$$z_k' = \frac{\alpha_k z_k + \beta_k}{\gamma_k z_k + \delta_k} \qquad (k = 1, 2, \cdots, n).$$

It is a theorem due to himself that a conclusion similar to that of Weierstrass can be drawn if the transformation just written is replaced by a projective transformation on the n variables z. This involves a conception of the points at infinity different from that of Weierstrass, and there are gradations between the two, with possibly similar theorems, if the variables are transformed one part projectively and the rest bilinearly. In Section 6 of this lecture Professor Osgood draws the very interesting conclusion that similar results can be obtained by assuming properties of the function f at a suitably characterized portion only of the region at infinity.

Let a power series in x, y be convergent for $|x| < r$, $|y| < s$, and let s be the largest positive constant for which, when r is fixed, the series has this convergence property. Then s is a function $s = \varphi(r)$ called the radius of convergence associated with r. Sections 7 and 8 of this lecture contain a summary of the known properties of the function $\varphi(r)$ and theorems related to it.

The last part of Lecture II describes generalizations of the famous theorems of Weierstrass and Mittag-Leffler concerning the construction of a function of a single complex variable having assigned poles and zeros in a given continuum. There always exists such a function analytic except at the poles, expressible as a quotient of two other functions each analytic at every point of the continuum. The essence of the generalization to functions of several complex variables lies in the characterization of the points which correspond to poles or zeros. Expressed very roughly the theorem is that if at every point $a = (a_1, a_2, \cdots, a_n)$ of the space a quotient

$$f_{(a)}(z_1, z_2, \cdots, z_n) = \frac{H_{(a)}(z_1, z_2, \cdots, z_n)}{G_{(a)}(z_1, z_2, \cdots, z_n)} \tag{2}$$

is assigned, where H and G are analytic and prime to each other at the point in question, then there exists a quotient $f = H/G$ of two permanently convergent power series which is equivalent at every point a to the corresponding quotient $f_{(a)}$. Two functions are equivalent at a point a if their quotients, taken both ways, have only removable singularities in a neighborhood $T_{(a)}$ of the point. Professor Osgood describes also a more general theorem of Cousin, and some consequences involving generalizations by Gronwall of the Weierstrassian notion of prime factors of integral functions.

The third lecture is concerned with singular points and analytic continuation. A non-essential singularity of the first kind, sometimes called a pole, is a point $(a_1, a_2, \cdots, a_n)$ at which the function is expressible in the form (2) with $H_{(a)} \neq 0$, $G_{(a)} = 0$ at $(a_1, a_2, \cdots, a_n)$; the point is a non-essential singularity of the second kind if $H_{(a)}$ and $G_{(a)}$ both vanish but have no vanishing factors in common there; all other non-removable singularities are essential.

Professor Osgood gives a number of theorems, by Kistler and Hartogs, specifying conditions under which a singularity can be removed after assigning suitably chosen values to the function at the point itself and at other exceptional points near it. No non-removable singularities are isolated when $n > 1$, and in particular every essential singularity is a limit point of others of the same type. Starting from a theorem concerning analytic continuation Professor Osgood shows how Hartogs has proved that if a function $f(x, y)$ of two complex variables is analytic except at the points of a two-dimensional continuous locus $y = \varphi(x)$ consisting of non-removable singularities, then $\varphi(x)$ itself must be analytic. This is a sharp contrast to the fact that the singularities of a function of a single complex variable may have a distribution as arbitrary as the boundary of an arbitrarily selected two-dimensional continuum.

Another striking theorem, deducible from a theorem of Levi which Professor Osgood characterizes as one of the most important contributions to the theory in recent years, is that a function $f(x, y)$ analytic (or meromorphic) at all points of the boundary of a finite and regular four-dimensional region T, can always be continued analytically (or meromorphically) throughout T. This has as a consequence that such a function can not have a finite lacunary region around which it is analytic. A three-dimensional locus

$$\varphi(x_1, x_2, y_1, y_2) = 0,$$

where $x = x_1 + ix_2$, $y = y_1 + iy_2$, can not be arbitrarily selected if it is to be a natural boundary for a function $f(x, y)$; it must satisfy a certain differential inequality. Levi-Civita has shown further that a function of two complex variables is in general uniquely determined if its values are assigned along a two-dimensional locus. But there are exceptional "characteristic" loci along which arbitrarily assigned values may determine none or an infinity of functions, a situation which reminds us forcibly that we are dealing more or less directly with initial values for solutions of a system of partial differential equations.

In Lecture IV Professor Osgood discusses certain questions in implicit function theory based upon the preparation theorem of Weierstrass. This theorem states that a function $F(u; x_1, x_2, \cdots, x_n)$ analytic at the origin and having a lowest term in the series $F(u; 0, 0, \cdots, 0)$ of order m, is expressible as a product

$$F = P\Omega, \tag{3}$$

where Ω is a series in $u, x_1, \cdots, x_n$ with constant term different from zero, while P is a polynomial in u of degree m with leading coefficient unity, its other coefficients being series in $x_1, \cdots, x_n$ vanishing at the origin. Near the origin the roots of F are clearly all roots of the polynomial P, and vice versa. The theorem mentioned in an earlier paragraph, which Professor Osgood stated tentatively in the Lectures, and later disproved by an example, would have implied a generalization of this including the case when $F(u; 0, 0, \cdots, 0)$ vanishes identically.

The preparation theorem gives information with respect to the roots of a single equation $F = 0$ in a neighborhood of the origin, which is completely satisfactory for many purposes. For a system of equations

$$\Phi_k(u_1, \cdots, u_l; x_1, \cdots, x_p) = 0 \quad (k = 1, 2, \cdots, l), \tag{4}$$

where the functions Φ_k are also analytic and vanish at the origin, Weierstrass has given a further important theorem, described in Section 6 of Lecture IV. He was interested in characterizing the manifold of $(l + p)$-dimensional points, rather than in the implicit functions u of the variables x defined by equations (4).

The relation between F and the polynomial P in equation (3) could equally well have been expressed in the form $P = MF$, where $M = 1/\Omega$ is a series with constant term different from zero. In a paper published in 1912* the author of this review has shown that when the homogeneous polynomials of lowest degree, the "characteristic" polynomials, of the series $\Phi_k(u_1, \cdots, u_l; 0, \cdots, 0)$, have a resultant R different from zero, there always exists a polynomial $P(u_l; x_1, \cdots, x_p)$ expressible linearly in the form

$$P = M_1\Phi_1 + M_2\Phi_2 + \cdots + M_l\Phi_l,$$

and such that every root system $(u_1, \cdots, u_l; x_1, \cdots, x_p)$ of equations (4) near the origin has values $(u_l; x_1, \cdots, x_p)$ making $P = 0$; while every sufficiently small root system of $P = 0$ is part of at least one solution of these equations near the origin in the $(l + p)$-dimensional space. The degree N of P in u_l is the product of the degrees of the characteristic polynomials. Further when the substitution

* A generalization of Weierstrass's preparation theorem for a power series in several variables, *Transactions Amer. Math. Society*, vol. 13 (1912), p. 133.

$$(5) \qquad u_l = - t_1 u_1 - t_2 u_2 - \cdots - t_{l-1} u_{l-1} - z$$

is made in equations (4), the polynomial corresponding to P has the form

$$Q(z;\, x_1, \cdots, x_p) = \prod_{i=1}^{N} (z - t_1 u_{i1} - t_2 u_{i2} - \cdots - t_{l-1} u_{i(l-1)} - u_{il}),$$

where the systems $(u_{i1}, \cdots, u_{il};\, x_1, \cdots, x_p)$ are the root systems of (4) corresponding to $x_1, \cdots, x_p$. The introduction of the variables t is a well-known device of the algebraic elimination theory serving to unite the u-values which belong to a single root system of the equations.

It seems to me that the theorems described by Professor Osgood in Section 7 of Lecture IV could be deduced at once from the properties of the polynomial Q. For example, since Q is of the Nth degree, the system (4) must have exactly N root systems near the ux-origin for each point $(x_1, \cdots, x_p)$ in a properly chosen neighborhood of the origin in the x-space. These systems will all be distinct unless $(x_1, \cdots, x_p)$ makes the z-discriminant of Q identically zero in the variables t. The coefficients of the various t-terms are, however, series in $x_1, \cdots, x_p$ which set equal to zero define a locus analogous to Professor Osgood's "locus D," but not, if I understand him correctly, exactly the same. At every point of this locus some of the root systems of equations (4) coincide. Professor Osgood has stated his results for a more general hypothesis than mine. He requires only that the series $\Phi_k(u_1, \cdots, u_l;\, 0, \cdots, 0)$ shall have no common root near the origin except that point itself. The theorems of my paper apply, however, under very general circumstances, since the degrees of the characteristic polynomials are entirely arbitrary and their coefficients subjected to an inequality only. I must confess that my proofs are complicated, and I hope that this undesirable feature will be eliminated when Professor Osgood publishes in detail the theorems in his more general form.

On page 196 of the Lectures there is a reference to my paper which might lead one to infer that its character is similar to that of a paper by MacMillan. The hypotheses of the two papers are the same, but their principal theorems are entirely different. MacMillan discusses the highly interesting theorem, originally stated by Poincaré, that the solutions of the equations (4) near the origin are the same as those of another system whose first members are polynomials in the variables u. The polynomials have in general extraneous roots which are not necessarily small. The polynomial Q described above defining the solutions near the origin would be found from them by eliminating the u's algebraically and applying the preparation theorem of Weierstrass to the resulting polynomial in z.

Lecture V describes a point of view in the algebraic function theory which seems most interesting and effective. Let $f(w, z) = 0$ be an irreducible algebraic equation of deficiency p, having a corresponding Riemann surface F' rendered simply connected by period cuts. When $p = 1$ the equation $t = w(z)$, where w is the integral of the first kind, maps F' upon a curvilinear parallelogram in the t-plane, and the theory of functions and integrals on the Riemann surface then corresponds completely to that of a system of elliptic and allied functions with specified periods and t as independent variable. The Weierstrassian function $\sigma(t)$ is the fundamental function for such a system, in terms of which all the others may be expressed. The theory is more uniform in the t-plane than on the Riemann surface, because the particular t-values which correspond to branch points of the surface are no more exceptional than other values of t.

For deficiencies $p > 1$ Klein has defined a "prime function" on the Riemann surface which plays a rôle analogous to that of the σ-function for the simpler case. Professor Osgood proposes to transfer the theory of this function to a t-plane on which the surface is mapped by a transformation $z = \varphi(t)$, the function φ being a suitably selected automorphic function. He shows concisely but clearly in this lecture how the various types of functions analogous to those on the Riemann surface may be expressed in terms of the prime function $\Omega(t, \tau)$, and points out the advantages, in many cases decisive, to be gained by this uniformization.

It is not to be expected that the complications of a structure like that of the algebraic function theory can be entirely eliminated by a process such as Professor Osgood would have us use. But it is possible, on the other hand, that they may be so localized as not to mar the more facile beauty of the other parts of the theory. A first difficulty from his point of view seems to lie in the proof of the existence of the uniformizing automorphic function $\varphi(t)$, a proof which he has treated in the second edition of his Funktionentheorie with detail. The prime function is defined in Lecture V as a limit

$$\Omega(t, \tau) = \lim_{\Delta t=0,\ \Delta\tau=0} \sqrt{\Delta t \Delta \tau e^{-\Pi_{t\tau}^{t+\Delta t,\, \tau+\Delta\tau}}},$$

where $\Pi_{t'\tau'}^{t\tau}$ is a function analogous to the difference

$$P_{\xi\eta}^{xy} = P_{\xi\eta}(x) - P_{\xi\eta}(y)$$

formed for a suitably chosen integral $P_{\xi\eta}(x)$ of the third kind on the Riemann surface. It is seen that the definition involves a function of four arguments, and it is on account of this circumstance that the fifth lecture is a natural continuation of the preceding four. For the theorems of Lecture II, in particular, describing the analytic character of a function of several simultaneous variables resulting from known properties of the function when all but one variable are considered as parameters, play an important and apparently indispensable rôle in establishing the existence and properties of the functions Π and Ω.

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SHORTER NOTICE.

Mathematiker-Anekdoten. By W. AHRENS. Band XVIII of the Mathematische Bibliothek edited by Lietzmann and Witting. Leipzig, B. G. Teubner, 1916. Pp. 56. Price 80 pfennige.

ANYTHING from the pen of Dr. Ahrens is always sure to be interesting. His Mathematische Unterhaltungen und Spiele, his Mathematische Spiele, and his Scherz und Ernst in der Mathematik are all well known and are everywhere appreciated. He is one of those rare writers on the bizarre in mathematics who keep their balance and turn out works which have the stamp of dignity and learning, while at the same time revealing the lighter side of the science. In other words, he is a man who evidently combines in his own soul the Scherz