THE TWENTY-THIRD SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The twenty-third summer meeting and eighth colloquium of the Society were held at Harvard University during the week September 4–8, 1916. Monday and Tuesday were devoted to the summer meeting proper, two sessions being held on each day for the presentation and discussion of papers. The colloquium opened on Wednesday morning and extended to Friday afternoon. An account of the colloquium appears in this number of the BULLETIN.

The following ninety-nine members attended the summer meeting:

Professor L. D. Ames, Professor R. C. Archibald, Professor G. N. Armstrong, Professor C. S. Atchison, Professor Clara L. Bacon, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor Henry Blumberg, Professor Maxime Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor W. D. Cairns, Professor B. H. Camp, Dr. A. B. Chace, Professor C. W. Cobb, Professor F. N. Cole, Professor L. L. Conant, Professor J. L. Coolidge, Dr. A. R. Crathorne, Professor Louise D. Cummings, Professor C. H. Currier, Professor D. R. Curtiss, Professor E. W. Davis, Dr. C. E. Dimick, Dr. C. R. Dines, Professor Arnold Dresden, Professor Otto Dunkel, Professor W. P. Durfee, Professor W. C. Eells, Professor John Eiesland, Professor L. P. Eisenhart, Professor G. C. Evans, Mr. G. W. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. L. R. Ford, Dr. M. G. Gaba, Professor W. C. Graustein, Dr. G. M. Green, Professor M. W. Haskell, Professor C. N. Haskins, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. M. Hoskins, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor Dunham Jackson, Professor W. W. Johnson, Dr. E. A. T. Kircher, Professor A. E. Landry, Mr. B. B. Libby, Dr. Joseph Lipka, Professor H. P. Manning, Professor J. L. Markley, Professor Helen A. Merrill, Dr. A. L. Miller, Professor H. B. Mitchell, Professor U. G. Mitchell, Professor C. L. E. Moore, Professor C. N. Moore, Professor G. D. Olds, Dr. G. A. Pfeiffer, Professor J. M. Poor, Professor L. H. Rice, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Dr. A. R. Schweitzer, Dr. Caroline E.

Seely, Dr. H. M. Sheffer, Dr. L. L. Silverman, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor R. B. Stone, Professor E. B. Stouffer, Professor H. W. Tyler, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor Roxana H. Vivian, Professor C. A. Waldo, Professor A. G. Webster, Dr. Mary E. Wells, Dr. C. E. Wilder, Professor F. B. Williams, Professor F. N. Willson, Dr. Euphemia R. Worthington, Mr. W. C. Wright, Professor C. H. Yeaton, Professor J. W. A. Young, Professor J. W. Young, Professor Alexander Ziwet.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Vice-Presidents E. R. Hedrick and Virgil Snyder. The Council announced the election of the following persons to membership in the Society: Mr. Herman Betz, Cornell University; Mr. J. A. Bigbee, High School, Little Rock, Ark.; Mr. Hillel Halperin, Vanderbilt University; Dr. J. R. Kline, University of Pennsylvania; Professor J. J. Luck, University of Virginia; Dr. F. J. McMackin, Dartmouth College. Seven applications for membership in the Society were received.

Through the generosity of Harvard University the Freshman Dormitories and dining room were thrown open for the use of the Society during the meeting. On Monday noon the members of the Society were shown the collection of mathematical models belonging to the University. On Wednesday afternoon a visit was paid to the University Library, and on Wednesday evening to the Observatory. Resolutions were adopted at the meeting expressing the thanks of the Society for the hospitality of the University and its officers.

Fraternal greetings were exchanged by cable with the Scandinavian mathematicians assembled at Stockholm. A vote of congratulation was tendered to the Secretary on his twenty-first year of service in that capacity.

The twenty-fifth anniversary of the broadening out of the Society into a national organization and the founding of the BULLETIN were celebrated at the banquet on Monday evening, at which eighty-four members and friends were present. Brief addresses were made by Professors Fiske, Johnson, Fine, Birkhoff, Hedrick, Webster, Coolidge, and the Secretary.

On Tuesday evening Professor D. E. Smith entertained the Society with an instructive account of "The relation of the history of economics to the history of arithmetic problems."

The following papers were read at this meeting:

(1) Professor J. C. FIELDS: "Direct derivation of the complementary theorem."

(2) Dr. C. A. FISCHER: "Note on the order of continuity of functions of lines."

(3) Dr. OLIVE C. HAZLETT: "On the theory of associative division algebras."

(4) Professor W. C. EELLS: "A statistical study of eminent mathematicians."

(5) Professor J. L. COOLIDGE: "The characteristic numbers of real algebraic plane curves."

(6) Dr. R. W. BURGESS: "The comparison of a certain case of the elastic curve with its approximation."

(7) Professor G. A. MILLER: "Orders of operators of congruence groups modulo $2^r 3^s$."

(8) Professor JOHN EIESLAND: "Sphere geometry (third paper)."

(9) Dr. A. J. KEMPNER: "Generalization of a theorem on transcendental numbers."

(10) Professor C. N. MOORE: "On the developments in Bessel's functions."

(11) Professor ARNOLD DRESDEN: "Supplementary note on the second derivatives of an extremal integral."

(12) Professor L. E. DICKSON: "An extension of the theory of numbers by means of correspondences between fields."

(13) Professor A. G. WEBSTER: "On a theory of acoustic horns."

(14) Professor E. H. MOORE: "On properly positive Hermitian matrices."

(15) Professor L. P. EISENHART: "Deformations of transformations of Ribaucour."

(16) Professors F. R. SHARPE and VIRGIL SNYDER: "On (2, 2) point correspondence between two planes."

(17) Professor F. H. SAFFORD: "Surfaces of revolution in the theory of Lamé's products."

(18) Professor DUNHAM JACKSON: "Note on the parametric representation of an arbitrary continuous curve."

(19) Professor DUNHAM JACKSON: "Note on representations of the partial sum of a Fourier series."

(20) Professor L. H. RICE: "Determinants of many dimensions."

(21) Dr. L. R. FORD: "Regular continued fractions."

(22) Professor M. W. HASKELL: "The eliminant of a system of forms."

(23) Professor E. V. HUNTINGTON: "A simple substitute for Duhamel's theorem."

(24) Professor E. V. HUNTINGTON and Dr. J. R. KLINE: "Sets of independent postulates for betweenness."

(25) Professor G. D. BIRKHOFF: "Dynamical systems with two degrees of freedom (second paper)."

(26) Professor E. B. VAN VLECK: "Non-loxodromic substitutions in n variables."

(27) Dr. L. I. HEWES: "Nomograms of adjustment."

(28) Mr. H. C. M. MORSE: "A theorem on the linear dependence of analytic functions of a single variable."

(29) Dr. G. A. PFEIFFER: "Note on the linear dependence of analytic functions."

(30) Dr. G. M. GREEN: "On the linear dependence of functions of one variable."

(31) Professor C. L. BOUTON: "Iteration and group theory."

(32) Dr. G. M. GREEN: "On the general theory of surfaces."

(33) Dr. W. V. GARRETSON: "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order. A particular solution."

(34) Dr. CAROLINE E. SEELY: "On series of biorthogonal functions."

(35) Professor A. B. FRIZELL: "Lemma for a new method of generating alephs."

(36) Professor JOHN EIESLAND: "Transformation theory of the flat complex and its associated line complex."

(37) Dr. A. R. SCHWEITZER: "On a type of quasi-transitive functional equations (second paper)."

(38) Dr. A. R. SCHWEITZER: "A problem in quasi-transitive functional equations."

(39) Dr. A. R. SCHWEITZER: "Some theorems on quasi-transitive functional equations."

(40) Dr. A. R. SCHWEITZER: "On the analogy between functional equations and geometric order relations."

(41) Dr. T. H. GRONWALL: "On the power series for log $(1 + z)$."

(42) Dr. T. H. Gronwall: "A problem in geometry connected with the analytic continuation of a power series."

(43) Dr. T. H. Gronwall: "On the convergence of Binet's factorial series for log $\Gamma(z)$ and $\psi(z)$."

(44) Dr. T. H. Gronwall: "On the zeros of the function $\beta(z)$ associated with the gamma function."

Mr. Morse was introduced by Professor Osgood. The papers of Professor Fields, Dr. Fischer, Professor Miller, Dr. Kempner, Professor Dickson, Professor E. H. Moore, Professor Safford, Dr. Hewes, Dr. Garretson, Dr. Seely, Professor Frizell, Dr. Schweitzer, and Dr. Gronwall were read by title. Abstracts of the papers appear below; the abstracts are numbered to correspond to the titles in the list above.

1. There is nothing in the nature of the complementary theorem which should require for its proof a preliminary determination of a formula for the actual number of the conditions imposed on a rational function of (z, u) of given form by a given set of orders of coincidence corresponding to a specific value of z. The proof here given by Professor Fields dispenses with the aid of such a formula and thus shortens by one step the approach to the theorem. The general complementary theorem then presents itself at a very early stage in the development of the theory of the algebraic functions, the only steps preliminary to its proof, besides those implied in the proof of the existence theorem for the branches of an algebraic function, being those involved in the determination of the reduced form for a rational function of (z, u), those involved in the proof of the integral character relative to the element $z - a$ of the principal coefficient of a rational function of (z, u) which is adjoint for the value $z = a$, and those involved in the proof of what we may call the theorem of principal residues which is derived therefrom and which is contained in the statement: The necessary and sufficient conditions in order that a rational function $H(z, u)$ may have orders of coincidence which are complementary adjoint to a given set of orders of coincidence for the value $z = a$ are obtained on equating to 0 the principal residue relative to $z = a$ in the product of $H(z, u)$ by the general rational function conditioned by the given set of orders of coincidence.

2. Dr. Fischer's paper appears in full in the present number of the Bulletin.

3. There is a famous theorem to the effect that the only linear associative algebras over the field of real numbers in which division is uniquely possible are the field of real numbers, the field of ordinary complex numbers, and real quaternions, the most recent of the numerous proofs of this being one by Professor Dickson. In his paper the theorem is a special case of a theorem for a certain class of algebras (called Type A) over a general algebraic field F which has a subalgebra S which can be exhibited as an algebra L over a field K with the r^2 units $i^s j^k$ $(s, k = 1, \cdots, r)$, where

$$(1) \qquad ji = \theta(i)j, \quad j^r = g(i),$$

where i is an element of S satisfying in K a uniserial abelian equation of degree r, with the roots $\theta(i), \cdots, \theta^r(i) = i$.

Dr. Hazlett's paper considers linear associative division algebras over a general algebraic field F, and finds that a necessary and sufficient condition that such an algebra be of Type A is that θ in (1) be the root of a certain algebraic equation, which we will call the Θ-equation. Then, from some properties of the Θ-equation, this paper proves that, if a linear associative division algebra over an algebraic field be of rank n, it is of order $m \cdot n$ where $m \leqq n$. Of this theorem, Frobenius's theorem is a corollary. It also follows that, if a linear associative division algebra, of rank n, over an algebraic field F, contain a number i which satisfies a uniserial abelian equation of degree n, then any number in the algebra is a polynomial in j with coefficients in $F(i)$, where (1) holds. There are numerous other theorems, some of which are generalizations of well known facts about quaternions.

4. From a representative group of standard histories of mathematics and of general encyclopedias, Professor Eells by the space method selects a group of one hundred most eminent mathematicians of history, discusses the reliability of the selection, and classifies and discusses the men selected according to nationality, period of history in which they flourished, principal branches of mathematics to which they contributed, and other features.

5. A real algebraic plane curve has two sorts of characteristic numbers. First, there are the total or Plücker characteristics, which are invariant for the general plane collineation. Sec-

ond, there are the real characteristics, as the number of real inflections, real double tangents, etc. These are invariant for real collineations. The only known identical equation connecting these latter is the equation of Klein. In Professor Coolidge's paper it is shown that, taking total and real characteristics together, the only universally valid equations are the equations of Plücker and Klein, and others deducible from them.

6. In the usual discussion of the elastic curve in elementary text-books of physics and engineering, a certain term in the differential equation is neglected and the curve obtained as the solution of this approximate equation is said to be a good substitute for the real elastic curve. In the case of a bow or a vertical column this approximate solution is a cosine curve, which is satisfactory if the given constants are the force and the deflection. In the more common problem, however, a bow or column of known length is bent by a known force. Dr. Burgess shows that if a formula for the deflection in terms of the force and the length is deduced from the cosine curve to apply to this case, the result is only about one half the true deflection as found from the real elastic curve.

7. The $\phi(m)$ positive integers which do not exceed the natural number m and are prime to m, when these numbers are combined by multiplication modulo m, constitute a group which is simply isomorphic with the group of isomorphisms of the cyclic group of order m. The principal object of Professor Miller's paper is to show how the order of each of the operators of this group can be directly obtained from the form of the corresponding number by means of general theorems relating to the automorphisms of an abelian group, whenever m is of the form 2^r3^s. Attention is called to the fact that in H. Weber's Lehrbuch der Algebra, second edition, volume 2 (1899), page 65, the exponent to which 5 belongs modulo 2^β is found by a very special method, and that a somewhat similar method is followed by P. Bachmann in the first part of his Niedere Zahlentheorie, 1902, page 331, while the method by means of the theorems noted above exhibits at once the exponent to which each number belongs.

8. Professor Eiesland's memoir is a continuation of a paper on the same subject read before the Society at the

meeting of December 28–29, 1915. An attempt has been made to build up a theory of flat complexes with a view to its application to sphere geometry. It has been shown that with the linear flat complex is associated a line complex which, in some important respects, is analogous to the null-system in ordinary space. The lines of this complex satisfy the system of Pfaffian equations

$$(1)\qquad \begin{array}{c} Qdx_i - Pdy_i + R_idz = 0, \\ R_{\frac{1}{2}(n-2)}dy_i - R_idy_{\frac{1}{2}(n-2)} = 0 \end{array} \qquad [i = 1, 2, \cdots, \tfrac{1}{2}(n-2)],$$

where

$$P = A_{\frac{1}{2}n}z + \Sigma A_kx_k - B_0, \quad Q = A_0z + \Sigma A_ky_k + B_{\frac{1}{2}n},$$
$$R_i = A_{\frac{1}{2}n}y_i - A_0x_i - B_i,$$

the coefficients A_i, B_i being the constants in the equation of the flat complex

$$(2)\qquad \Sigma A_i\rho_i + \Sigma B_i\sigma_i = 0.$$

All the lines of the line complex which pass through a point form a flat pencil, i. e., they lie in a 2-flat. The relation between point and 2-flat is involutory. Every line is the axis of $\infty^{\frac{1}{2}(n-4)}$ flats of the complex (2).

9. The generalized theorem of Dr. Kempner's paper is as follows:

I. An infinite series

$$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a_n}$$

represents a transcendental number, provided

(1) $\gamma_n = p_n/q_n$ is a real rational number; p_n and q_n are either finite or grow infinite with n in some particular manner; it is for example amply sufficient if $|p_n|$, $|q_n|$ are both $< R^n$, R any fixed positive number;

(2) each denominator a_n is a positive integer and a factor of the next denominator a_{n+1};

$$(3)\qquad \lim_{n=\infty} \frac{a_{n+1}}{a_n^2} \geqq 1$$

if the limit exists; if it does not exist, then

$$\liminf_{n=\infty} \frac{a_{n+1}}{a_n^2} \geqq 1.$$

In 1915 (see BULLETIN, March, 1916, page 285) the author presented the theorem

I'. $$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a^{c^n}} \cdot x^n$$

represents a transcendental number, provided
(1') γ_n restricted as above under (1), but
(2') only a finite number of the $\gamma_n = 0$,
(3') a and c any integers $\geqq 2$,
(4') x any rational number $\neq 0$.
At the April, 1916, meeting at Chicago, Professor H. Blumberg announced that he had succeeded, by a modification of the author's method of proof, in entirely removing the condition (2'). Neither Professor Blumberg's proof nor the author's proof of I' makes essential use of the assumption that the denominators be of the form a^{c^n}, and analysis of the two proofs shows that it is sufficient to assume that the denominators satisfy (2) and (3) of the generalized theorem.

In Theorem I, nothing is gained by considering the series

$$\sum_{n=1}^{n=\infty} \frac{\gamma_n}{a_n} \cdot x^n$$

for rational x, since this is included in I. Theorem I could not be proved before Professor Blumberg's welcome contribution had eliminated (2').

10. Professor Moore's paper appeared in full in the October BULLETIN.

11. In a paper presented to the Society at its meeting in Chicago, December, 1914, Professor Dresden gave formulas for the second derivatives of the extremal integrals arising in the theories of the integrals

$$\int f(x; y_1, \cdots, y_n; y_1', \cdots, y_n')dx$$

and

$$\int f(y_1, \cdots, y_n; y_1', \cdots, y_n')dt.$$

A certain lack of symmetry in the formulas for the second of these theories made them rather unsuitable for applications and undesirable from a theoretical point of view. By using the "normal" solutions of the system of Jacobi differential equations recently introduced by Professor Bliss, it has been found possible to remove this lack of symmetry and to simplify the formulas very materially. It is the object of the present note to derive these simplified formulas.

12. Professor Dickson's paper will appear in full in an early number of the BULLETIN.

13. Since no boundary problem for an unclosed non-plane surface has been solved with the exception of Kelvin's spherical bowl, there is as yet no theory of horns of finite length. As such a theory is eminently desirable not only for musical instruments but for reception apparatus, including the phonograph, megaphone, and a great variety of acoustical instruments, Professor Webster has been driven to a very coarse approximate theory, which works surprisingly well in practice. Two assumptions are made, first that the cross-section of the horn is infinitesimal with respect to the wave length, and varies as any given function of the distance from a certain section; second that at the open end Helmholtz's assumption for tubes of infinitesimal cross-section may be used. The first assumption leads to a linear differential equation, and when the section varies as a power of the axial distance this may be integrated by Bessel functions of fractional index. The cases of cylinders, cones, and hyperbolic horns such as are used in the orchestra come under this case. The assumption for the end, even when the diameter is half the wave length instead of an infinitesimal part of it, is found to agree within a few per cent with experimental determinations. Any horn is characterized by four constants, and may be replaced by any other horn whatever having the same constants.

14. The new theory of linear integral equations in general analysis, of which a special instance is Hilbert's theory of limited symmetric bilinear forms in infinitely many variables, under development by Professor Moore, is based on a properly positive Hermitian matrix or numerically valued function ϵ or $\epsilon(s, t)$ of two variables s, t ranging independently over a general class $\mathbf{P}$, viz., a function ϵ of such a nature that (1) $\overline{\epsilon(s, t)} = \epsilon(t, s)$ for every s, t of $\mathbf{P}$; (2) every finite principal minor is positive, i. e., for every set $(p_1, p_2, \cdots, p_n)$ of a

finite number n of distinct elements of $\mathbf{P}$, the determinant of the n^2 functional values $\epsilon(p_i, p_j)(i, j = 1, 2, \cdots, n)$ is positive. For the Hilbert instance, the class $\mathbf{P}$ is the class of positive integers, and the function ϵ is δ, $\delta(s, t) = d_{st} = 1(s = t)$, $0(s \neq t)$.

The present paper of Professor Moore presents various absolute and relative theorems of existence of such matrices ϵ. As an example, if $\mathbf{P}$ is the class of all real valued functions p or $p(x)$ on the interval 01, ϵ or $\epsilon(s, t) = \exp \int_0^1 s(x)t(x)dx$ is such a matrix ϵ. This example is of especial importance in that for the corresponding instance of the general theory we meet for the first time a non-trivial instance of an integration process whose integrand functions are themselves functional operations; this "integration over a function space" is secured in virtue of the fact that the integration process involves in its definition no assumed metrical features of the range of integration.

15. When a system of spheres involves two parameters, their envelope consists, in general, of two sheets, say Σ and Σ_1, and the centers of the spheres lie upon a surface S. A correspondence between Σ and Σ_1 is established by making correspond the points of contact on the same sphere. When the lines of curvature on Σ and Σ_1 correspond, Σ_1 is said to be in the relation of a transformation of Ribaucour with Σ, and vice versa. It is a known property of envelopes of spheres that if the surface of centers S be deformed and the spheres be carried along in the deformation, the points of contact of the spheres with their envelope in the new position are the same as before deformation. Professor Eisenhart determines the transformations of Ribaucour which in deformations of the surface of centers S become transformations of Ribaucour. The surfaces Σ in this case have the same spherical representation of their lines of curvature as isothermic surfaces. The paper will be published in the October number of the *Transactions*.

16. Marletta (*Palermo Rendiconti,* volume 17 (1903)) has discussed the (2, 2) correspondence between the points of two planes in which a line is transformed into a conic or a cubic. In the paper of Professors Sharpe and Snyder the two images

of a point in one plane are the intersections of a corresponding line and conic in the other. The image of a line is a quintic of genus 2. The correspondences are classified according to the number of basis points the nets of conics and the quadratic systems of lines may have.

The results obtained by Marletta and those known earlier all appear as particular cases.

17. In the investigation of surfaces of revolution in the theory of Lamé's products it is necessary to find a function of two variables such that a certain known function of the original one shall be the sum of two functions, one of the first variable and the other of the second. After the original function has been determined, the resolution of the known function into a sum as stated has required separate treatment for each case considered. In this paper Professor Safford has obtained the general expression for the component parts of the sum in terms of the original function. The meridian curves are in general Weierstrass $\wp$-function curves. The paper will be sent to the *Archiv der Mathematik und Physik*, in which several related papers have already appeared.

18. A continuous curve may be defined by a pair of equations

$$x = f(t), \qquad y = \varphi(t),$$

where the functions f and φ are defined and continuous throughout some interval, and are not both constant throughout that interval. For some discussions it is convenient to suppose further that they are not simultaneously constant throughout any sub-interval of their interval of definition. The question arises, whether this assumption implies a restriction on the curve itself, that is, on the set of points (x, y) given by the equations, or whether it is a restriction merely on the particular parametric representation employed. It is obvious that the latter is the case, if the number of intervals of constancy is finite.

It was proved by Fréchet (*Rendiconti Palermo*, volume 22) that this remains true, even when there are infinitely many of the intervals in question. Professor Jackson gives a different proof of the same theorem.

19. In the general theory of the convergence of expansions in series associated with ordinary linear differential equations,

as developed by Birkhoff (*Transactions,* 1908) and others, the sum of the first n terms of the series is represented by a contour integral in the plane of the complex parameter. The contour integral, in the fundamental particular case of Fourier's series, must of course be identical in value with the ordinary real formula for the partial sum of the series. In his second note, Professor Jackson enumerates the steps in the process of verifying this identity by direct evaluation of the contour integral, without going back to the individual terms which make up the sum. The Fourier's series can be derived either from a system of the first order or from one of the second order, and the work is carried through for both cases.

20. An extended definition of a determinant is given by Professor Rice, which applies to determinants of more than three dimensions and enables us to remove the restriction in Cayley's law of multiplication and to set up a new case in Scott's law of multiplication. New formulas are obtained for the known process of decomposition of a determinant into determinants of fewer dimensions, and a new process called crossed decomposition is described. Fresh light is thrown upon the function known as a "determinant-permanent," a limitation hitherto thought necessary being done away. Finally a generalization to p dimensions is made of Metzler's theorem in two dimensions concerning a determinant each of whose elements is the product of k factors.

21. A continued fraction

$$b_0 \pm \frac{a_1}{b_1 \pm} \frac{a_2}{b_2 \pm} \cdots \frac{a_n}{b_n \pm} \cdots$$

is called regular if $a_i = \pm 1$ or $\pm \sqrt{-1}$, and

$$b_i = m_i + n_i \sqrt{-1},$$

where m_i and n_i are real integers. In Dr. Ford's paper the classic pentahedral division of half-space corresponding to the group of Picard is employed to give a geometrical interpretation of the convergents of the continued fraction.

It is shown how any such continued fraction can be determined by means of a curve lying in the half-space. Explicit formulas are given for setting up the continued fraction corresponding to a given curve. By a study of the generating curve general theorems are established relative to (1) the value of the convergents as approximations to the sum of the continued fraction, and (2) the periodicity of the continued fraction.

22. Professor Haskell gives a method of constructing the eliminant of a system of forms which leads to a direct proof of the well-known theorem that the number of solutions of a system of simultaneous equations is equal to the product of their degrees. Beginning with a system of linear forms, he uses the theory of symmetric functions to construct a series of eliminants, in which the linear forms are, one after the other, replaced by forms of higher degree, and thus arrives at the contravariant which factors into the linear factors corresponding to the various solutions of the system of forms of any degree, and without any extraneous factor.

23. Professor Huntington's note contains the following substitute for Duhamel's theorem, which may seem simpler than the substitutes already published in the *Annals of Mathematics* by Osgood, R. L. Moore, and Bliss: Suppose that a required quantity P is associated with a real interval, $x = a$ to $x = b$, in a way that suggests dividing the interval into n small parts, or elements, and regarding P as the sum of n separate contributions, one from each element. Suppose also that a set of one or more functions, $f(x)$, $F(x)$, $\cdots$, can be found such that no matter what the value of n, the contribution of a typical element, from $x = x$ to $x = x + \Delta x$, can be expressed "approximately" in the form $[f(x)\cdot F(x)\cdots]\Delta x$—that is, so that the exact value of the contribution lies between $[f_1\cdot F_1\cdots]\Delta x$ and $[f_2\cdot F_2\cdots]\Delta x$, where f_1, F_1, $\cdots$ are the least, and f_2, F_2, $\cdots$ the greatest values of $f(x)$, $F(x)$, $\cdots$ in the interval in question. Then the required quantity P can be computed as a definite integral: $P = \int_a^b [f(x)\cdot F(x)\cdots]\,dx$, provided only that the functions involved are continuous from $x = a$ to $x = b$.

24. The paper of Professor Huntington and Dr. Kline starts with the following basic list of twelve postulates for be-

tweenness: (A) If AXB then BXA. (B) If A, B, C are distinct, then at least one of the relations BAC, CAB, ABC, CBA, ACB, BCA will be true. (C) AXY and AYX cannot both be true. (D) If ABC is true, then the elements A, B, C are distinct. Further, if A, B, X and Y are distinct, then: (1) If XAB and ABY then XAY; (2) If XAB and AYB then XAY; (3) If XAB and AYB then XYB; (4) If AXB and AYB then AXY or AYX; (5) If AXB and AYB then AXY or YXB; (6) If XAB and YAB then XYB or YXB; (7) If XAB and YAB then XYA or YXA; (8) If XAB and YAB then XYA or YXB. The question then is: Given any subset S of the twelve postulates of this basic list, and any postulate P of the list, not belonging to S; is P deducible from S? An exhaustive answer to this question is given in 71 theorems of deducibility and 68 theorems of non-deducibility, these latter being established by the aid of 44 examples of pseudo-betweenness. It is found that eleven different sets of independent postulates may be selected from the basic list, as follows: 1, 2; 1, 5; 1, 6; 1, 7; 1, 8; 2, 4; 2, 5; 3, 5; 3, 4, 6; 3, 4, 7; 3, 4, 8, with the addition of postulates A, B, C, D in each of these sets. The paper will be offered to the *Transactions*.

25. Professor Birkhoff proves two theorems concerning the invariant points of a surface of any genus p under a one-to-one transformation into itself. The second of these theorems, in which it is assumed that the transformation leaves an area integral invariant, is based upon a modification of Poincaré's last geometric theorem. The results admit of application to dynamical systems with two degrees of freedom.

26. At the last meeting of the Society in Chicago Professor Van Vleck presented a paper on the "Composition of non-loxodromic linear substitutions, $z' = (az + b)/(cz+d)$." When expressed in unimodular form

$$z_1' = az_1 + bz_2, \quad z_2' = cz_1 + dz_2$$

with unit determinant, the non-loxodromic substitution has a real characteristic equation. Professor Van Vleck defined correspondingly a non-loxodromic linear substitution in n homogeneous variables to be one which, expressed in unimodular form, has a real characteristic equation. He then announced his first results in extension of the preceding paper, for proof of which new methods were found necessary.

27. D'Ocagne has shown that an equation of the form

$$(1)\qquad \begin{vmatrix} f_{12} & g_{12} & h_{12} \\ f_{34} & g_{34} & h_{34} \\ f_{56} & g_{56} & h_{56} \end{vmatrix} = 0,$$

where f_{ij}, g_{ij}, h_{ij} are functions of the same two variables, may be represented by a nomogram with at most three curve nets. It is proposed by Dr. Hewes to generalize the form of equation (1) by the use of the equation

$$(2)\qquad \begin{vmatrix} f_{ij} & g_{ij} & h_{ij} \\ f_{kl} & g_{kl} & h_{kl} \\ f_{mn} & g_{mn} & h_{mn} \end{vmatrix} = 0,$$

where the subscripts may take on the values 0, 1, 2, $\cdots$, 6. There result nomograms requiring rotation of the index and of wide application.

28. Mr. Morse proves the following theorem, which is due to Professor Osgood:

Let $f_1(t), f_2(t), \cdots, f_p(t)$ be p functions of the complex variable t, analytic in a region S of the t-plane. Then a necessary and sufficient condition for the linear dependence of these functions is the identical vanishing of a p-rowed determinant whose ith row ($i = 1, 2, \cdots, p$) is $f_i^{(\lambda_1)}(t_1), f_i^{(\lambda_1-1)}(t_1), \cdots, f_i'(t_1), f_i(t_1), f_i^{(\lambda_2)}(t_2), f_i^{(\lambda_2-1)}(t_2), \cdots, f_i'(t_2), f_i(t_2), \cdots, \cdots, f_i^{(\lambda_\mu)}(t_\mu), f_i^{(\lambda_\mu-1)}(t_\mu), \cdots, f_i'(t_\mu), f_i(t_\mu)$, where $t_1, t_2, \cdots, t_\mu$ are independent variables and the upper indices denote differentiation.

The theorem includes the ordinary wronskian theorem for analytic functions as a special case, and the method of proof affords a new demonstration for the latter theorem.

29. Dr. Pfeiffer proves the theorem given by Mr. Morse by showing that successive differentiation of the determinant in question, and subsequent equating of all the independent variables, reduces the determinant to the wronskian of the given functions.

30. In his first paper Dr. Green establishes a condition under which the vanishing of the determinant of the two preceding papers is sufficient for the linear dependence of the functions involved, these functions not being required to be analytic. The existence of only those derivatives which appear in the said determinant is required, the only restriction being the non-vanishing of a certain first minor in the determinant. The proof is analogous to the one given by Bôcher* for the corresponding wronskian theorem.

31. Professor Bouton's note gives a discussion of a method of finding all the continuous one-parameter groups which contain a given individual point transformation in n variables. This given transformation is taken in the form $x_i' = x_i + \psi_i(x)$ where the ψ_i are convergent power series beginning with terms of the second degree. In the course of the work the result of iterating the given transformation k times is derived.

32. In extending theorems of metric differential geometry to projective, it is necessary to substitute for the congruence of normals to a surface a congruence which is likewise uniquely determined by the surface, but projectively. Wilczynski's directrix congruence of the second kind might be made to serve the purpose; but this congruence has not the peculiar projective property of the normal congruence, whose developables intersect the surface in a conjugate net. In his second paper, Dr. Green defines and characterizes geometrically a congruence which has this desired property. In a previous paper† he defined a relation R, by means of which to every line through a point P of the surface is made to correspond a unique line lying in the tangent plane, and vice versa—the correspondence depending upon the parametric net. Let the surface be referred to its asymptotic curves, and let l be a line in the tangent plane, l' the corresponding line through the point P. Project the asymptotic curves on the tangent plane from any point on l', and construct the conics C_1 and C_2 which osculate these projections at the point P. Then there exists but one pair of lines l, l' such that the intersec-

* "Certain cases in which the vanishing of the wronskian is a sufficient condition for linear dependence," *Transactions Amer. Math. Society*, vol. 2 (1901), pp. 139–149, Theorem II.

† "On rectilinear congruences and nets of curves on a surface." Abstract in the BULLETIN, vol. 22 (1916), p. 274.

tions of l with the asymptotic tangents are the double points of the involution determined by the two pairs of points in which l cuts the conics C_1 and C_2. Let L_1 and L_2 be the intersections of the asymptotic tangents with the line l thus uniquely determined, and denote by D_1 and D_2 the intersections of the asymptotic tangents with the directrix of the first kind. Calling P_1 the harmonic conjugate of D_1 with respect to P and L_1, and P_2 the harmonic conjugate of D_2 with respect to P and L_2, one thus obtains a line m joining the points P_1 and P_2 and lying in the tangent plane to the surface. Corresponding to the line m, in the relation R, there exists a line m' passing through the point P. For every point P of the surface exists such a line m', and the congruence of these lines has the property that its developables cut the surface in a conjugate net.

Among other results arrived at may be mentioned a geometric characterization of the tetrahedron of reference which gives rise to a certain canonical development of the non-homogeneous coordinates of a point of the surface. This development was given by Darboux without any geometric characterization. The points L_1 and L_2 of the preceding investigation are two of the vertices of the tetrahedron, and the line m' is an edge.

33. Dr. Garretson's paper treats of the irregular integrals of the non-homogeneous linear equation of any order, where the roots of the characteristic equation are distinct. The irregular point is taken as the point at infinity. Two theorems given by Dini for the homogeneous equation have been generalized so as to apply to the non-homogeneous equation and combined in one theorem. To this end certain ideas given by Horn have been employed. Use is also made of the solutions obtained by Professor C. E. Love for the homogeneous equation. The paper leads to the asymptotic development, in the Poincaré sense, of a particular solution of the non-homogeneous equation.

34. A necessary and sufficient condition for the essentially uniform convergence of series of orthogonal functions in two variables

$$\sum_{i=1}^{\infty} \frac{\varphi_i(x)\varphi_i(y)}{\lambda_i}$$

has been given by Lauricella. Miss Seely considers the corresponding problem for series of the form

$$\sum_{i=1}^{\infty} \frac{\varphi_i(x)\psi_i(y)}{\lambda_i},$$

where the φ's and ψ's are a biorthogonal system of functions, and obtains conditions of which Lauricella's theorem is a special case.

35. In the November BULLETIN* Professor Frizell used a simple special case of the following lemma. Given a triply infinite partition $u_{\lambda\mu\nu}$ of an infinite set of symbols v_ι selected from an ω-series a_i ($i, \iota, \lambda, \mu, \nu = 1, 2, \cdots$) and a non-enumerable well-ordered set P of different permutations among the members of an ω-series; if now the elements $u_{\lambda\mu\nu}$ are exhibited in a rectangular array of infinite series, wherein λ, μ denote the row and column respectively:

$$\begin{array}{llllllll} u_{111}, & u_{112}, & \cdots; & u_{121}, & u_{122}, & \cdots; & u_{131}, & u_{132}, \ \cdots; \ \cdots \\ u_{211}, & u_{212}, & \cdots; & u_{221}, & u_{222}, & \cdots; & u_{231}, & u_{232}, \ \cdots; \ \cdots \\ u_{311}, & u_{312}, & \cdots; & u_{321}, & u_{322}, & \cdots; & u_{331}, & u_{332}, \ \cdots; \ \cdots \\ u_{411}, & u_{412}, & \cdots; & u_{421}, & u_{422}, & \cdots; & u_{431}, & u_{432}, \ \cdots; \ \cdots \\ & \cdots & & & \cdots & & & \cdots \end{array}$$

then the permutations P operating on the indices ν (each pair λ, μ being held fast) produce new matrices $\mathfrak{M}$ yielding permutations $\mathfrak{P}$ which can not be put into one-to-one correspondence with the set P. The present note is devoted to the proof of the general proposition.

36. To the conformal group in S_{n-1} (n even) corresponds in $\bar{S}_{n-1}$ a group of contact transformations $G_{\frac{1}{2}(n+1)(n+2)}$ which transforms the ∞^n flats

$$(1) \qquad \begin{gathered} x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d, \\ az = \Sigma c_i x_i + ad - \Sigma c_i b_i \end{gathered}$$

inter se. The group also preserves asymptotic lines on a surface. An important subgroup of this group is considered by Professor Eiesland, viz.:

* Vol. 22, No. 2, p. 73.

1. The group of the complex $a + d = 0$ which carries null-flats into null-flats, with which is closely connected
2. The group of the associated line complex, and also
3. The group of parameters which shows how the flats (1) are transformed when the surface elements of $\bar{S}_{n-1}$ are subjected to the transformations of $G_{\frac{1}{2}(n+1)(n+2)}$. This group is the group of the quadric surface

$$e = ad - \Sigma b_i c_i$$

so that the geometry of flats is equivalent to a geometry on a quadric surface in a space of $n + 1$ dimensions.

The following theorem is also proved:

It is always possible by means of a linear transformation to reduce the system of Pfaffian equations

$$Qdx_i - Pdy_i + R_i dz = 0, \quad R_{\frac{1}{2}(n-2)}dy_i - R_i dy_{\frac{1}{2}(n-2)} = 0$$

to any one of the following forms:

(1) $dx_i - zdy_i + y_i dz = 0, \quad y_{\frac{1}{2}(n-2)}dy_i - y_i dy_{\frac{1}{2}(n-2)} = 0,$

or

(2) $zdx_i - x_i dz = 0, \quad x_i dy_{\frac{1}{2}(n-2)} - x_{\frac{1}{2}(n-2)}dy_i = 0,$

or

(3) $dx_i = 0.$

37. In a former paper Dr. Schweitzer considered quasi-transitive functional equations of the type

(1) $f(u_1, u_2, \cdots, u_{n+1}) = f(y_1, y_2, \cdots, y_{n+1}),$

where

$$u_i = f(t_1, t_2, \cdots, t_n, x_i), \quad y_i = \sum_{k=1}^{n+1} m_{ik}x_k,$$

and it was shown that if

$$f(x_1, x_2, \cdots, x_{n+1}) = \sum_{k=1}^{n+1} \alpha_k x_k$$

then

$$\sum_{k=1}^{n+1} \alpha_k = 0$$

and $\alpha_2, \alpha_3, \cdots, \alpha_{n+1}$ respectively satisfy n algebraic equations of the nth degree with coefficients in the field of

$$M_{st} = m_{st} - m_{1t} \quad (s, t = 2, 3, \cdots, n + 1).$$

Each of the roots α is, moreover, a rational function of α_{n+1} with regard to M_{st} $(s, t \neq 1)$.

In the present paper, the problem of deriving conditions on the M_{st} such that the n equations of the nth degree possess the same roots $\alpha_2, \alpha_3, \cdots, \alpha_{n+1}$ is discussed. In this case, the resulting single equation, if irreducible in the field of certain of the M's in which the coefficients of the equation lie, is regular with reference to this field. Conversely, if the n relations (derived from the equation (1)) in terms of $\alpha_2, \cdots, \alpha_{n+1}$ and rational with regard to M_{st}, are subjected to the substitutions of a regular group on the symbols $\alpha_2, \cdots, \alpha_{n+1}$, then from the resulting n sets of n relations it follows that $\alpha_2, \cdots, \alpha_{n+1}$ are roots of the same equation. This consideration leads to sufficient conditions for the desired properties of the $\alpha_2, \cdots, \alpha_{n+1}$. For example, when $n + 1 = 3, 4, 5$ the author obtains four theorems corresponding to the four regular groups on two, three and four symbols:

I. If

$$f(u_1, u_2, u_3) = f(-bx_2, -bx_3, -ax_1 + (a - b)x_3)$$

then α_2, α_3 are roots of the quadratic $\xi^2 - a\xi + ab = 0$.

II. If

$$f(u_1, u_2, u_3, u_4) = f(y_1, y_2, y_3, y_4),$$
$$f(v_1, v_3, v_4, v_2) = f(y_1, y_3, y_4, y_2),$$

where

$$v_i = f(t_1, t_2, x_i, t_3)$$

then $\alpha_2, \alpha_3, \alpha_4$ are roots of the cubic $\xi^3 - a\xi^2 + ab\xi - ac = 0$ provided

$$\sum_{t=1}^{4} M_{st} = 0 \ (s = 2, 3, 4), \quad M_{21} = 0, \quad M_{31} = 0,$$
$$M_{22} = M_{43}, \quad M_{33} = M_{42}, \quad M_{23} = M_{32},$$
$$M_{22} + M_{33} + M_{44} = a, \quad M_{22} + M_{33} + M_{23} = b,$$
$$M_{22}M_{33} - M_{23}^2 = c, \quad M_{23}M_{44} = c - (M_{22}^2 + M_{33}^2).$$

III. If

$$f(u_1, u_2, u_3, u_4, u_5) = f(y_1, y_2, y_3, y_4, y_5),$$
$$f(v_1, v_3, v_2, v_5, v_4) = f(y_1, y_3, y_2, y_5, y_4),$$
$$f(w_1, w_4, w_5, w_2, w_3) = f(y_1, y_4, y_5, y_2, y_3),$$

where

$$v_i = f(t_1, t_2, t_3, x_i, t_4), \quad w_i = f(t_1, t_2, x_i, t_3, t_4)$$

then $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ are roots of the biquadratic $\xi^4 - a\xi^3 + ab\xi^2 - ac\xi + ad = 0$ provided

$$\sum_{t=1}^{5} M_{st} = 0 \ (s = 2, 3, 4, 5), \quad M_{21} = 0, \quad M_{31} = 0,$$

$$M_{41} = 0, \quad M_{22} = M_{52}, \quad M_{33} = M_{53}, \quad M_{44} = M_{54},$$

$$M_{32} = M_{42}, \quad M_{23} = M_{43}, \quad M_{24} = M_{34},$$

$$M_{22} + M_{33} + M_{44} + M_{55} = a$$

and M_{55}, b, c, d are certain rational functions of M_{22}, M_{33}, M_{44}, M_{23}, M_{32}, M_{24}.

IV. This theorem is analogous to theorem III. Presumably one obtains for $n + 1 > 5$ theorems on algebraic equations corresponding to every regular substitution group on n symbols.

38. Dr. Schweitzer proposes the following problem: Let

$$(1) \qquad f(u_1, u_2, \cdots, u_{n+1}) = \phi(x_1, x_2, \cdots, x_{n+1}),$$

where

$$u_i = f(t_1, t_2, \cdots, t_n, x_i) \quad (i = 1, 2, \cdots, n + 1);$$

to determine the function f under the following conditions: If

$$f(x_1, x_2, \cdots, x_{n+1}) = f_0(x_1, x_2, \cdots, x_{n+1})$$

is a solution of (1) so also is

$$f(x_1, x_2, \cdots, x_{n+1}) = f_0(x_1, \cdots, x_k, x_{i_{k+1}}, \cdots, x_{i_{k+j}}, x_{k+j+1}, \cdots, x_{n+1}),$$

where $i_{k+1}, \cdots, i_{k+j}$ each range over $k + 1, k + 2, \cdots, k + j$ and are distinct and

$$\begin{pmatrix} (k+1), & \cdots, & (k+j) \\ i_{k+1}, & \cdots, & i_{k+j} \end{pmatrix}$$

is any given substitution of any given group G on the symbols $x_{k+1}, \cdots, x_{k+j}$. Two important special cases of this problem are (I) the determination of a class of solutions which are permuted among themselves when subjected to the substitutions of the group G when the latter is supposed, e. g., regular; (II) the determination of a solution of (1) which is invariant under the substitutions of G. The linear functions discussed in the preceding paper of the author yield solutions of problems of type (I), whereas corresponding to (II) the following theorem is proved: Given any substitution group G

$$G: \qquad \left\{\begin{matrix} X_2, & \cdots, & X_n \\ X_{i_2}, & \cdots, & X_{i_n} \end{matrix}\right\} (X_{n+1})$$

there exists a group of quasi-transitive functional equations, namely,

$$E \left\{\begin{matrix} 1, & 2, & 3, & \cdots, & n, & n+1 \\ 1, & i_2, & i_3, & \cdots, & i_n, & n+1 \end{matrix}\right\}$$

simply isomorphic with G and which have as a common particular solution a function which is formally invariant under the substitutions of the group G.

39. Dr. Schweitzer proves the following theorems:

I. If

$$f\{z, f(x, y)\} = f\{y, f(x, z)\}, \quad \phi\{f(x, y), x\} = \psi(y),$$
$$f\{\phi(x, y), x\} = \psi(y), \quad f\{\psi(x), \psi(y)\} = f(y, x),$$

then there exists $\chi(x)$ such that

$$\chi f(x, y) = \chi(x) - \chi(y), \quad \chi\phi(x, y) = \chi(x) - \chi(y) + c,$$
$$\chi\psi(x) = c - \chi(x)$$

where c is an arbitrary constant. Therefore

$$\phi\{\phi(x, y), \phi(x, z)\} = \phi(z, y)$$

and

$$f\{f(x, y), f(x, z)\} = f(z, y).$$

II. If

$$f\{u_1, u_2, \cdots, u_{n+1}\} = f\{\theta(x_1), \cdots, \theta(x_{n+1})\}$$

where

$$u_i = f\{t_1, t_2, \cdots, t_n, x_i\} \quad (i = 1, 2, \cdots, n+1)$$

then

$$\psi f(x_1, \cdots, x_{n+1}) = \Theta\{\psi(x_1) - \psi(x_2), \cdots, \psi(x_1) - \psi(x_n)\}$$
$$+ k(\psi(x_i) - \psi(x_{n+1})) + c_1,$$
$$\psi\theta(x) = - k\psi(x) + c_2$$

where k, c_1, and c_2 are arbitrary constants and Θ is an "arbitrary" function of $(n - 1)$ variables.

III. If

$$f\{u_1, u_2, \cdots, u_{n+1}\}$$
$$= f\{\alpha_1(x_1), \alpha_2(x_3), \alpha_3(x_4), \cdots, \alpha_n(x_{n+1}), \alpha_{n+1}(x_2)\}$$

then

$$\psi f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} a_i \psi(x_i) + l,$$

$$\psi \alpha_i(x) = m_i \psi(x) + p_i \quad (i = 1, 2, \cdots, n+1),$$

where

$$\sum_{i=1}^{n+1} a_i p_i = 0, \quad \sum_{i=1}^{n+1} a_i = 0, \quad a_2 = m_{n+1}, \quad a_{n+1} = m_1,$$

$$a_{j+1} = \frac{m_2 m_3 \cdots m_j}{m_1^{j-1}} \cdot m_{n+1} \quad (j = 2, 3, \cdots, n-1)$$

and $m_1^n = m_2 m_3 \cdots m_n m_{n+1}$. The m's and p's are constants arbitrary apart from the conditions indicated.

40. The object of Dr. Schweitzer's fourth note is to point out an analogy that exists between geometric order relations in the foundations of geometry and functional equations. This analogy consists in interpreting functionally abstractions of geometric order relations; in other words, it is shown that certain geometric order relations and functional equations give rise to the same general abstract relations resembling the "associations" of Grassmann and Pietzker. The geometric order relations are based* mainly on the author's "right-handedness" relation $\alpha R_n \beta_1, \beta_2, \cdots, \beta_n$ and "interior" relation $\alpha I_n \beta_1 \beta_2 \cdots \beta_{n+1}$ and the functional equations arrived at belong to categories previously defined by the author, especially the "quasi-transitive" functional equations. The general logical theory of relations relevant to the preceding position is that of Veronese.

41. Dr. Gronwall proves by actual analytic continuation of the power series $\log(1 + z) = z - z^2/2 + z^3/3 - \cdots$ the familiar facts that $\log(1 + z)$ has its only finite singular point at $z = -1$, and that the continuation along a closed path winding once in the positive direction around $z = -1$ has the affect of increasing $\log(1 + z)$ by $2\pi i$.

* Cf. *Amer. Jour. of Mathematics*, 1909, pp. 365–410.

42. Dr. Gronwall solves the following problem: given a power series in z defining an analytic function with the sole singularity $z = -1$ at finite distance, what is the minimum number m of intermediate points $a_1, a_2, \cdots, a_m$ needed to effect the analytic continuation of the given power series to a given point z_0? Writing $z = -1 + re^{\theta i}$, $-\pi \leqq \theta \leqq \pi$, it is found that the set of points z_0, for which no more than m intermediate points are necessary, is formed by the interior of the curve

$$r = \left(2 \cos \frac{\theta}{m+1}\right)^{m+1}, \quad (-\pi \leqq \theta \leqq \pi).$$

The totality of possible positions of the last intermediate point a_m for a given z_0 is also determined.

43. The four series in question are known to converge absolutely and uniformly for $R(z) \geqq \epsilon > 0$, but diverge for $R(z) < 0$ ($R(z)$ = real part of z, and ϵ arbitrarily small). Dr. Gronwall investigates the convergence on the boundary line $R(z) = 0$, and finds that all four series converge uniformly for $R(z) \geqq 0$, $|z| \geqq \epsilon$; two of the series are absolutely convergent in the same region, while the remaining two do not converge absolutely for any purely imaginary value of z.

44. In his Handbuch der Gammafunktion, Nielsen shows that the function

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z}{2}\right) + \psi\left(\frac{1-z}{2}\right)\right],$$

where $\psi(z) = d \log \Gamma(z)/dz$, has no real zeros, and raises the question whether any complex zeros exist. In the present paper, Dr. Gronwall establishes the existence of an infinity of such zeros, and gives their asymptotic expressions.

F. N. Cole,
Secretary.

THE CAMBRIDGE COLLOQUIUM.

The eighth colloquium of the American Mathematical Society was held in connection with its twenty-third summer meeting at Harvard University, Cambridge, Massachusetts. At the April, 1915, meeting of the Council the invitation of