42. Dr. Gronwall solves the following problem: given a power series in z defining an analytic function with the sole singularity $z = -1$ at finite distance, what is the minimum number m of intermediate points $a_1, a_2, \cdots, a_m$ needed to effect the analytic continuation of the given power series to a given point z_0? Writing $z = -1 + re^{\theta i}$, $-\pi \leqq \theta \leqq \pi$, it is found that the set of points z_0, for which no more than m intermediate points are necessary, is formed by the interior of the curve

$$r = \left(2 \cos \frac{\theta}{m+1}\right)^{m+1}, \quad (-\pi \leqq \theta \leqq \pi).$$

The totality of possible positions of the last intermediate point a_m for a given z_0 is also determined.

43. The four series in question are known to converge absolutely and uniformly for $R(z) \geqq \epsilon > 0$, but diverge for $R(z) < 0$ ($R(z)$ = real part of z, and ϵ arbitrarily small). Dr. Gronwall investigates the convergence on the boundary line $R(z) = 0$, and finds that all four series converge uniformly for $R(z) \geqq 0$, $|z| \geqq \epsilon$; two of the series are absolutely convergent in the same region, while the remaining two do not converge absolutely for any purely imaginary value of z.

44. In his Handbuch der Gammafunktion, Nielsen shows that the function

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z}{2}\right) + \psi\left(\frac{1-z}{2}\right)\right],$$

where $\psi(z) = d \log \Gamma(z)/dz$, has no real zeros, and raises the question whether any complex zeros exist. In the present paper, Dr. Gronwall establishes the existence of an infinity of such zeros, and gives their asymptotic expressions.

F. N. Cole,
Secretary.

THE CAMBRIDGE COLLOQUIUM.

The eighth colloquium of the American Mathematical Society was held in connection with its twenty-third summer meeting at Harvard University, Cambridge, Massachusetts. At the April, 1915, meeting of the Council the invitation of the Division of Mathematics of Harvard University to hold the summer meeting and colloquium at Harvard was accepted, and a committee of arrangements appointed, consisting of Professors Osgood, Bôcher, E. H. Moore, P. F. Smith, and the Secretary. The courses of lectures were announced in the preliminary circular of April, 1916, and printed syllabi were distributed at the meeting. The colloquium opened Wednesday morning, September 6, and continued until Friday afternoon; three lectures were delivered on each of the first two days, and four on Friday. A number of books of reference were placed on the reserved shelves of the university library for the use of those attending the colloquium, and ample opportunity for conferences was provided in Smith and Standish Halls throughout the week. The following sixty-nine persons were in attendance, a number considerably exceeding that of any previous colloquium.

Professor L. D. Ames, Professor R. C. Archibald, Professor Clara L. Bacon, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor H. Blumberg, Professor M. Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. R. W. Burgess, Dr. A. B. Chase, Professor C. W. Cobb, Professor F. N. Cole, Professor J. L. Coolidge, Dr. A. R. Crathorne, Professor Louise D. Cummings, Professor C. H. Currier, Miss A. M. Curtis, Dr. C. E. Dimick, Professor A. Dresden, Professor Otto Dunkel, Professor J. Eiesland, Professor G. C. Evans, Mr. G. W. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. L. R. Ford, Dr. M. G. Gaba, Professor W. C. Graustein, Dr. G. M. Green, Professor M. W. Haskell, Professor C. N. Haskins, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Mr. M. T. Hu, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor D. Jackson, Mr. R. Keffer, Dr. E. A. T. Kircher, Professor A. E. Landry, Mr. F. W. Loomis, Professor J. L. Markley, Professor Helen A. Merrill, Dr. A. L. Miller, Professor H. B. Mitchell, Professor C. N. Moore, Dr. F. D. Murnaghan, Dr. F. H. Murray, Dr. G. A. Pfeiffer, Dr. T. A. Pierce, Professor R. G. D. Richardson, Dr. A. R. Schweitzer, Dr. Caroline E. Seely, Dr. L. L. Silverman, Professor Clara E. Smith, Professor V. Snyder, Professor R. P. Stephens, Professor R. B. Stone, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor A. G. Webster, Dr. Mary E. Wells, Dr. C. E. Wilder, Dr. Euphemia R. Worthington, Professor J. W. Young, Professor Alexander Ziwet.

Two courses of five lectures each were given:

I. Professor G. C. Evans: "Topics from the theory and applications of functionals, including integral equations."

II. Professor O. Veblen: "Analysis situs."

Abstracts of the lectures follow below. The lectures will soon be published in full as Volume V of the Colloquium Series.

I.

Lecture I. Functionals of Curves and Surfaces in Three Dimensions; Variational Equations.

1. Additive and non-additive functionals of curves in two and three dimensions; existence of a derivative; relation to functions of point sets.
2. The vector of a functional.
3. Variation of a functional:

$$\delta\Phi[c] = \int_c \Phi'[c|s]\delta n(s)ds.$$

$$\delta\Phi'[c|s] = \int_c \Phi''[c|s_1, s]\delta n(s_1)ds_1 + a_0(s)\delta n(s) + \sum_1^n a_i(s)\delta n^{(i)}(s).$$

$$\delta\Phi[c] = \int_c [V_x(x, y, z)dydz + V_y(x, y, z)dzdx + V_z(x, y, z)dxdy].$$

4. Functionals of surfaces; exceptional points and curves.
5. Variational equations (equations in functional derivatives of form $\Phi'[c|x, y] = F[c|\Phi, x, y]$); the equation for the Green's function; Hadamard's equation.
6. The condition for integrability; self-adjointness in the second variation.
7. Partial equations and the theory of characteristics.
8. Relation of variational equations to integro-differential equations.

Lecture II. Complex Functionals in Spaces of Three and Four Dimensions.

1. Complex additive functionals of curves; integrability; isogeneity, $\Phi_1 + i\Phi_2$ being isogenous to $F_1 + iF_2$; conjugateness in the ellipse belonging to the vectors V_1 of F_1 and V_2 of F_2; equations which vectors of Φ_1 and Φ_2 must independently satisfy.

2. Case where V_1 and V_2 are unit vectors parallel to x and y axes respectively.

3. The invariant $H[\Phi, \Phi']$; extension of Green's theorem; theorems on determinateness; special case when the triple product $[V_1 V_2 ds] = 0$ represents an integrable equation; reduction by a transformation to case where V_1 and V_2 are unit vectors parallel to x and y axes.

4. Generalization of Cauchy's theorem for the integral of a function of a complex variable; conjugateness; non-additive functionals.

5. Complex functionals of surfaces in space of four dimensions; integrability; isogeneity; integrals of analytic functions of two complex variables.

Lecture III. Implicit Functional Equations.

1. Volterra's theorem; calculation of the variation; reduction of the implicit functional equation to a linear integral equation.

2. Application to existence theorems.

3. Linear functional as limit of an integral; as Stieltjes integral; as Lebesgue integral.

4. Special forms of the linear integral equation with a parameter; the equations of the third and first kinds.

Lecture IV. Integro-Differential Equations of Bôcher Type.

1. Hypothetical experiments as basis of physics.

2. Bôcher's treatment of Laplace's equation; generalization to curvilinear coordinates; in three dimensions; Poisson's equation; Cauchy and Weierstrass regions.

3. The equations

$$\int_c \lambda \frac{\partial u}{\partial n} ds = 0, \qquad \int_c \lambda \frac{\partial u}{\partial n} ds = \int\int f(x, y) d\sigma;$$

principal solution.

4. Extension of Green's theorem.

5. Calculation of the Green's function for elliptic and parabolic equations; the Green's function and Green's theorem for integro-differential equations of Volterra type, where the variables are not separable.

6. Generalization of hyperbolic differential equation; stair solutions; non-homogeneous equation.

Lecture V. Direct Generalizations of the Theory of Integral Equations.

1. Moore's general analysis; bases and closure properties; generalization of Fredholm theory, of Hilbert-Schmidt theory; mixed linear equations.

2. Commutativity; associative combination; permutability.

3. The complex operation; convergence properties; differential properties.

4. Volterra's theory of permutable functions; fundamental rôle of the Volterra relation; fractions of composition; logarithms of composition; symbolical treatment of integro-differential equations.

II.

Lecture I. One-Dimensional Analysis Situs.

1. Analysis situs has two chief divisions, dealing (1) with continuity considerations and (2) with combinatorial ones. These lectures are to deal primarily with (2), making enough use of (1) to give substance to (2).

2. The 0-cell, 0-dimensional manifold (or point-pair), 1-cell, 1-dimensional manifold (or curve or circle) and the 1-dimensional complex, C_1, or linear graph.

3. Separation and sense in a 1-cell and a circle. Continuous transformation and deformation. Theorems on point sets, iteration, etc.

4. The matrix $\| \eta_{ij} \|$ of a C_1. The equations (modulo 2) determined by this matrix and the interpretation of their solutions as circuits in C_1.

5. If C_1 is connected there exists a linearly independent set of $\alpha_1 - \alpha_0 + 1$ simple circuits on which all others are linearly dependent. If $\alpha_1 - \alpha_0 + 1 = 0$, the graph is a tree.

6. Introduction of sense-relations, the matrix $\| \epsilon_{ij} \|$, and the corresponding equations. The number of solutions in a complete linearly independent set is again $\alpha_1 - \alpha_0 + 1$.

7. Notice of special investigations of linear graphs.

Lecture II. The n-Dimensional Complex.

8. The n-cell, n-dimensional sphere, n-dimensional complex C_n. Closed and open or bounded complexes, n-dimensional circuits, n-dimensional manifolds M_n.

9. The matrices $\| \eta_{jk}^1 \| = \mathrm{H}_1, \cdots, \| \eta_{jk}^n \| = \mathrm{H}_n$ of a C_n.

The sets of equations (modulo 2) determined by $H_1, \cdots, H_n$ and the interpretation of their solutions as circuits. $H_i \cdot H_{i+1} = 0$ $(i = 1, \cdots, n-1)$.

10. Definition of the numbers $R_0, R_1, \cdots, R_n$.

$$\alpha_0 - \alpha_1 + \cdots + (-1)^n \alpha_n$$
$$= 1 + (-1)^n + (R_0 - 1) + \cdots + (-1)^n \cdot (R_n - 1).$$

If C_n determines a manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=1}^{n-1} (-1)^i (R_i - 1).$$

11. Sense relations and the Poincaré matrices $\| \epsilon_{jk}^1 \| = E_1, \cdots, \| \epsilon_{jk}^n \| = E_n$. Discrimination between one and two-sided complexes. The sets of linear equations corresponding to $E_1, \cdots, E_n$. Congruences and homologies. $E_i \cdot E_{i+1} = 0$ $(i = 1, 2, \cdots, n-1)$.

12. The Betti numbers $P_0, P_1, \cdots, P_n$ as defined by Poincaré. For any complex,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=0}^{n} (-1)^i (P_i - 1).$$

For a two-sided manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + (-1)^n + \sum_{i=1}^{n-1} (-1)^i (P_i - 1).$$

For a one-sided manifold,

$$\sum_{i=0}^{n} (-1)^i \alpha_i = 1 + \sum_{i=1}^{n-1} (-1)^i (P_i - 1).$$

13. Theorems on matrices whose elements are integers. Reduction of $E_1, \cdots, E_n$ to normal form. Poincaré's coefficients of torsion. Relation between $R_0, \cdots, R_n, P_0, \cdots, P_n$, and the coefficients of torsion.

Lecture III. Manifolds of n Dimensions.

14. Singular complexes in a manifold M_n. Every closed C_k in an n-cell bounds a C_{k+1}.

15. The numbers R_i and P_i and the coefficients of torsion are the same for all complexes C_n into which an M_n may be subdivided, and hence are the same for all manifolds with

which M_n is in $(1-1)$ continuous reciprocal correspondence. Alexander's proof.

16. Dual complexes. Duality relation satisfied by the constants R_i for all manifolds and by P_i and the coefficients of torsion for two-sided manifolds. The characteristic, $\sum_{i=0}^{n}(-1)^i\alpha_i$, is zero if n is odd.

17. One-to-one correspondence between manifolds. Homeomorphism, homoömorphism, and internal transformation. Application of the theorems of separation of an n-cell.

18. Combinatorial proof of the theorems of separation of an n-cell by polyhedra. The theorem of Jordan and its generalizations.

Lecture IV. Two-Dimensional Manifolds.

19. Reductions of the complex defining a two-dimensional manifold. Normal forms and classification of surfaces.

20. The group of a two-dimensional manifold. Representation in the parabolic and hyperbolic planes.

21. Transformations and equivalence of curves on a surface. Homotopy and isotopy.

22. Continuous transformations and deformations. Theorems of Tietze.

Lecture V. The Group of a Complex.

23. The group defined by any complex or manifold.

24. Generating operations and essential relations. Theorem of Tietze.

25. The Cayley color diagram as developed by Dehn.

26. The group of a knot according to Dehn.

27. Relation to the older theories of knots.

Lecture VI. Three-Dimensional Manifolds.

28. Reductions of the complex defining a three-dimensional manifold.

29. A three-dimensional manifold as one cell with a singular boundary.

30. The Heegard diagram.

31. Any manifold may be decomposed into four cells without singularities.

32. The problem of classifying three-dimensional manifolds.
33. Particular three-dimensional manifolds. Riemann spaces. Poincaré spaces.

VIRGIL SNYDER.

NOTE ON THE ORDER OF CONTINUITY OF FUNCTIONS OF LINES.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, September 4, 1916.)

IT has been proved that if a linear function of a line has continuity of the zeroth order, it can be expressed as the limit of a sequence of definite integrals,* as an integral of Stieltjes,† or as a Lebesgue integral.‡ As has been remarked by Bliss,§ many of the functions occurring in the calculus of variations do not have such continuity. The object of the present note is to show that if a function $U[y(x)]$ is linear and has continuity of the nth order, and if $y(x)$ is of class $C^{(n)}$, then $U[y(x)]$ is equal to the sum of a linear function of $d^n y(x)/dx^n$ which has continuity of the zeroth order, and a function of the values of $y(x)$ and its derivatives at an end point of the curve considered.

The proof is very simple. If $y(x)$ is of class $C^{(n)}$ it can be expressed as

$$y(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!}(x-a)^{n-i}.$$

Then, since $U[y(x)]$, is linear,

$$U\left[\overset{b}{\underset{a}{y(x)}} \right] = V\left[\overset{b}{\underset{a}{y^{(n)}(x)}} \right] + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!} U\left[\overset{b}{\underset{a}{(x-a)^{n-i}}} \right],$$

where the function

$$V[y^{(n)}(x)] = U\left[\int_a^x dx_1 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n \right]$$

* Hadamard, "Leçons sur le Calcul des Variations," p. 299.

† Riesz, *Annales Scientifiques de L'Ecole Normale Supérieure*, vol. 28 (1911), p. 43.

‡ Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 140.

§ Bliss, *Proc. Nat. Acad. Sciences*, vol. 1 (1915), p. 173.