The work closes with a lecture treating "The notion of cause, with applications to the free-will problem." The discussion is very acute, the finest in the book. Hume's classical analysis is resumed and shown to require extension and this is undertaken. The vulgar notion of a cause as compelling its effect must be abandoned as having no warrant in logic and no essential rôle in natural science. A cause may as well accompany or follow its effect as precede it. Far more important than the notion of cause is that of causal law, which has been above stated in one form. It is acutely contended that, as causes do not compel, the acts of will may be caused without being externally coerced, and that omiscience, including knowledge of the entire future, is consistent with every thing in freedom that is worth preserving.

The book as a whole must be judged as an important contribution to the science of philosophy even if the reader must remain convinced that much that is destined to continue to be called philosophy will not, through logical analysis or other means, yield solid, scientific, objective knowledge. Personal idiosyncrasies are themselves facts and they are often more interesting than, and quite as important as, generic results that ignore them. In the future, as in the past, the value of philosophy will consist, not wholly in propositions established by it, but largely in philosophizing. Let the two kinds flourish side by side, but let them not be confounded.

CASSIUS J. KEYSER.

SHORTER NOTICES.

Combinatory Analysis. By Major PERCY A. MACMAHON. Volume 1. Cambridge University Press, 1915. 300 + xix pp.

THE author states that "the object of this work is, in the main, to present to mathematicians an account of theorems in combinatory analysis which are of a perfectly general character, and to show the connection between them by as far as possible bringing them together as parts of a general doctrine. Little attempt has been hitherto made either to make a general attack upon the territory to be won or to coordinate and arrange the ground that has been already gained. The combinatory analysis as considered in this work occupies the ground between algebra, properly so called, and the higher arithmetic. The methods employed are distinctly algebraical and not arithmetical."

Section I is devoted to the study of symmetric functions. The theory is developed with particular reference to the theory of the partitions of numbers. In general we may have in view the symmetric function

$$\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_s^{p_s},$$

the whole function consisting of a number of terms similar to the one attached to the sign of summation; in each term we may take $p_1, p_2, \cdots, p_s$ to be in descending order of magnitude, but the quantities α, therein appearing, will be any s selected from $\alpha_1, \alpha_2, \cdots, \alpha_n$ in any permutation. It thus appears that the succession of numbers $p_1, p_2, \cdots, p_s$ in descending order of magnitude is a sufficient specification of the symmetric function and, if

$$p_1 + p_2 + \cdots + p_s = w,$$

we may regard the function as denoted by the partition $(p_1p_2\cdots p_s)$ of the number w. It is now shown that the general theory of combinations is essentially involved in the algebra of monomial symmetric functions. Every multiplication of monomial symmetric functions involves a theorem in combinations.

Further developments of the theory of symmetric functions are made by introducing the two differential operators of Hammond:

$$d_\lambda = \frac{d}{da_\lambda} + a_1\frac{d}{da_{\lambda+1}} + a_2\frac{d}{da_{\lambda+2}} + \cdots$$

$$D_s = \frac{1}{s!}(d_1^s),$$

where (d_1^s) indicates a symbolic multiplication as usual in Taylor's expansion. In Chapters IV and V it is shown that the algebra of these operators is parallel with the algebra of symmetric functions.

Section II is a generalization of the theory of Section I. The theory of symmetric functions is associated not with a number but with the partition of a number. The theory of Section I is that particular part of a more general theory which is associated with that partition of a number which is composed wholly of units. Hammond's operators d and D are given an extended field of operation and, in the enlarged theory, are shown to be important instruments for multiplying and evaluating the symmetric functions that present themselves. In Chapter IV binomial coefficients are treated as symmetric functions denoted by partitions with zero parts, and the formation of symmetrical tables involving them is explained.

In Section III the enumeration of combinations and permutations is treated from the point of view supplied by the theory of symmetric functions. In Chapter II a theorem is established termed "a Master Theorem from the masterly and rapid fashion in which it deals with various questions otherwise troublesome to solve. Many illustrations of its power are given. In particular in Chapter III it is shown to supply instantly the solution of the generalized 'problème des rencontres.' In finding expressions for the sum of powers of binomial coefficients it is singularly effective."

Chapter V is devoted to lattice permutations. Consider an assemblage of letters $\alpha^p\beta^q\gamma^r\cdots$ in which the numbers $p, q, r, \cdots$ are in descending order of magnitude. This particular permutation of the assemblage can be denoted by a regular graph consisting of rows of dots. The successive rows will have $p, q, r, \cdots$ dots respectively and the graph is the same as serves to denote the partition $(pqr\cdots)$ of the number $p+q+r+\cdots$. Such a graph may be

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & & & & & \end{array}$$

for $p = 6, q = 4, r = 1$.

If we take *any* permutation of $\alpha^p\beta^q\gamma^r\cdots$ we shall arrive finally at the same graph by proceeding from left to right of the permutation and placing a dot in the first row, or in the second, or in the third according as we reach a letter α, β, or γ, etc.

The author states that lattice permutations "are in themselves interesting but are of importance principally because they are shown in Volume II to supply the key to solutions of certain questions which involve arrangements, under given conditions, of numbers in spaces of two and three dimensions; numbers not arranged at points in a straight line but at the crossing points of lines which compose lattices in two and three dimensions."

Section IV is devoted to the theory of the compositions of numbers. The word connotes a partition in which account is taken of the order of occurrence of the parts. Thus the partition a, b, c of the number $a + b + c$ would have six compositions *abc*, *acb*, *bac*, *bca*, *cab*, *cba* involving the parts a, b, c. Chapter I treats with unipartite numbers; Chapter II with multipartite numbers, a graphical representation being given; Chapter III with the graphical representation of the compositions of tripartite and multipartite numbers. Chapter IV gives a complete solution of Simon Newcomb's problem: A pack of cards of any specification is taken—say there are p cards marked 1, q cards 2, r cards 3, and so on—and being shuffled is dealt out on a table; so long as the cards that appear have numbers that are in ascending order of magnitude, equality of numbers counting as ascending order, they are placed together in one pack, but directly the ascending order is broken a fresh pack is commenced and so on until all the cards have been dealt. The probability that there will result exactly m packs or at most m packs is required. Chapter V gives a generalization of the foregoing theory.

In Section V the subject of the perfect partitions of numbers is dealt with as a necessary preliminary to the discussion of arrangements upon a chess board. The latter, as involving conditions to be satisfied by the numbers appearing in a row or in a column, are of the magic square nature. Operators and functions are designed with the object of discovering the laws of formation of the systems of diagrams to which they naturally lead. As the simplest example choose the operation d/dx and the function x^n. We observe that if x^n be written out in the form $xxx \cdots x$ the operation is equivalent to writing unity for one of the x's in all possible ways and adding the results together. We obtain nx^{n-1} because one x can be selected in n different ways. So operating n times successively with d/dx we dissect the operation into $n!$ distinct operations and these give rise to $n!$ distinct diagrams. For consider a square of n^2 compartments; we may place a unit at the intersection of the rth row and cth column to indicate that when d/dx was operating for the rth time our process was to substitute unity for the cth x counting from the left. So we obtain $n!$ diagrams which possess the property that n units appear one in each row and one in each column. We thus by the operation upon the function enumerate the diagrams with this property.

The problem of the Latin square is solved. This problem is to place n different letters $a, b, c, \cdots$ in each row of a square of n^2 compartments in such wise that, one letter being in each compartment, each column involves the whole of the letters. The number of arrangements is required. "The question is famous because, from the time of Euler to that of Cayley inclusive, its solution was regarded as being beyond the powers of mathematical analysis. It is solved without difficulty by the method of differential operators of which we are speaking. In fact it is one of the simplest examples of the method which is shown to be capable of solving questions of a much more recondite character." Most of the operators are those of the infinitesimal calculus and the numbers involved in the partitions of the functions are positive integers, excluding zero. If we admit zero as a part in the partitions, we have to do with the operations of the calculus of finite differences.

In Section VI the theory of distributions is applied to the enumeration of the partitions of multipartite numbers.

The author states that "In the present volume there appears a certain amount of original matter which has not before been published. It involves the author's preliminary researches in combinatory theory which have been carried out during the last thirty years."

The book is not intended for class room work. There is an absence of problems, but plenty of exercise for one who carries out in detail all of the work indicated. There is no index but a full table of contents. There are errors. It is not our intention to catalogue these. We refer to one merely to support our statement: on page 5 last line for a_2^p write $a_2^{p_2}$. At certain places conditions are implied which we think should be explicitly stated. We mention but one: page 4 line 3 we should have $\lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots + s\lambda_s = s$. Problems concerning symmetry of determinants find no place in the present treatise. There are thirteen pages of tables at the end exhibiting in a condensed form some of the results of the foregoing theory. In general the work is well written and clear.

W. V. LOVITT.