which satisfy $T(x, y) = 0$). No curves represented by equations of the form $\varphi_1(x, y) = 0$, $\varphi_2(x, y) = 0$ can intersect on the sine curve or the tangent curve excepting at the origin, nor on the exponential curve $x = e^y$ excepting in the point (1, 0).

2. The system $\varphi_1(x, y) + p(t)\varphi_2(x, y) = 0$, $\varphi_3(x, y) = 0$, $\varphi_4(x, y) = 0$ can have no simultaneous solutions except those, if any, which simultaneously satisfy $\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 0$, where $p(t)$ is a polynomial in t, a transcendental number.

3. All the singularities of the curves represented by $\psi(x, y) \equiv \varphi_1(x, y) + t\varphi_2(x, y) = 0$ which require $\partial\psi/\partial x = 0$ and $\partial\psi/\partial y = 0$, lie upon $\varphi_1(x, y) = 0$ and $\varphi_2(x, y) = 0$ if there are any at all.

4. All singularities of $\varphi_0(x, y) + p_1(t)\varphi_1(x, y) + \cdots + p_n(t)\varphi_n(x, y) = 0$ must lie upon each of the curves represented by the equations $\varphi = 0$, $\varphi_1 = 0$, $\cdots$, $\varphi_n = 0$ where $p_i(t)$, $(i = 1, \cdots, n)$, are polynomials in t, a transcendental number, the coefficients of the polynomials being algebraic numbers.

13. In this paper, Professor Bennett examines by elementary methods the form of a closed algebraic correspondence upon an algebraic curve or Riemann surface. As a result of certain elementary properties of involutions, and the group properties of closed algebraic and finite systems of points, the relations between closed correspondences and variable inscribed plane configurations are described. A systematic method of securing fundamentally different generalizations of the closure problems arising in connection with Poncelet polygons is obtained.

ARNOLD DRESDEN,
Secretary of the Section.

THE TWENTY-THIRD ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-third annual meeting of the Society, which was held in New York City on Wednesday and Thursday, December 27–28, 1916, was in several respects an exceptional occasion. It took place in the midst of the convocation week series of meetings of the American association for the advancement of science and its long train of affiliated societies; and was immediately followed by the second annual meeting of the newly organized Mathematical association of America, with which the Society has not only a large common membership but also a general community of interest highly beneficial to both. The annual meeting is always one of the largest of the year, being the season of the election of officers and other members of the Council and the transaction of important business. This year it was especially marked by the delivery of the retiring address of President E. W. Brown, the subject of which was "The relations of mathematics to the natural sciences." This was presented before a joint session of the Society with the Mathematical association, the Astronomical society of America, and Section A of the American association and was followed by the retiring address of Vice-President A. O. Leuschner of Section A, on "Derivation of orbits—theory and practice." A joint dinner of the four organizations was held on Thursday evening at the Park Avenue Hotel, with an attendance of 143 members and friends. Much of the credit for the great success of the meetings is due to the joint committee on arrangements and to the programme committees of the Mathematical association.

Under all these favorable circumstances the attendance at the four sessions of the Society exceeded all previous records. It included the following 131 members and perhaps others whom the Secretary failed to identify in the multitude:

Professor R. B. Allen, Professor R. C. Archibald, Professor C. H. Ashton, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor R. D. Beetle, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor J. W. Bradshaw, Professor E. W. Brown, Dr. T. H. Brown, Professor Daniel Buchanan, Professor W. G. Bullard, Dr. R. W. Burgess, Professor W. D. Cairns, Professor Florian Cajori, Professor W. B. Carver, Professor A. B. Coble, Professor Abraham Cohen, Professor F. N. Cole, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Professor C. H. Currier, Professor T. W. Edmondson, Professor L. P. Eisenhart, Professor T. C. Esty, Professor G. C. Evans, Mr. G. W. Evans, Professor F. C. Ferry, Professor J. C. Fields, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor T. M. Focke, Professor W. B. Ford, Professor A. S. Gale, Professor W. A. Garrison, Professor O. E. Glenn, Mr. T. E. Gravatt, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. V. Gummere, Dr. W. L. Hart, Dr. Mary G. Haseman, Professor M. W. Haskell, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Dr. A. A. Himwich, President C. S. Howe, Professor L. A. Howland, Professor L. S. Hulburt, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. E. A. T. Kircher, Dr. J. R. Kline, Mr. E. H. Koch, Jr., Professor J. K. Lamond, Mr. Harry Langman, Professor Florence P. Lewis, Professor G. H. Ling, Dr. Joseph Lipka, Professor W. R. Longley, Dr. J. V. McKelvey, Professor Emilie N. Martin, Professor Helen A. Merrill, Dr. Mansfield Merriman, Professor E. J. Miles, Dr. A. L. Miller, Professor Bessie I. Miller, Professor G. A. Miller, Professor H. B. Mitchell, Professor H. H. Mitchell, Professor R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor Richard Morris, Mr. G. W. Mullins, Professor G. D. Olds, Professor F. W. Owens, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Professor J. M. Poor, Professor H. W. Reddick, Professor R. G. D. Richardson, Mr. J. F. Ritt, Professor E. D. Roe, Jr., Professor R. E. Root, Dr. J. T. Rorer, Professor D. A. Rothrock, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Mary E. Sinclair, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Professor Virgil Snyder, Professor Pauline Sperry, Dr. J. M. Stetson, Professor Henry Taber, Professor H. D. Thompson, Dr. J. I. Tracey, Professor C. B. Upton, Professor J. N. Van der Vries, Mr. H. S. Vandiver, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Dr. J. H. Weaver, Mr. H. E. Webb, Professor A. G. Webster, Dr. Mary E. Wells, Professor H. S. White, Professor E. E. Whitford, Mr. J. K. Whittemore, Professor A. H. Wilson, Professor E. B. Wilson, Professor Ruth G. Wood, President R. S. Woodward, Professor J. W. Young, Dr. Mabel M. Young.

President Brown occupied the chair, being relieved by Vice-Presidents Hedrick and Snyder and by Professor Olds. The Council announced the election of the following persons to membership in the Society: Professor H. H. Conwell, University of Idaho; Mr. Robert Dysart, Boston, Mass.; Dr. Mary G. Haseman, Johns Hopkins University; Mr. J. B.

Scarborough, North Carolina Agricultural and Mechanical College; Mr. J. J. Tanzola, U. S. Naval Academy. Ten applications for membership in the Society were received.

In response to an invitation received from the Department of Mathematics of the University of Chicago, it was decided to hold the summer meeting and colloquium of the Society at that university in 1919. A committee was appointed to arrange for the summer meeting of 1917. A committee was also appointed to proceed with the early publication of the Cambridge Colloquium Lectures; the volume will probably appear in the early summer.

The total membership of the Society is now 732, including 75 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 490; the number of papers read was 205. The number of members attending at least one meeting during the year was 278. At the annual meeting 235 votes were cast. The Treasurer's report shows a balance of $10,198.38, including the life membership fund of $6,039.87. Sales of the Society's publications during the year amounted to $1,434.28. The Library now contains 5,377 volumes, excluding unbound dissertations.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

President,	Professor L. E. Dickson.
Vice-Presidents,	Professor A. B. Coble, Professor E. B. Wilson.
Secretary,	Professor F. N. Cole.
Treasurer,	Professor J. H. Tanner.
Librarian,	Professor D. E. Smith.

Committee of Publication,

Professor F. N. Cole,
Professor Virgil Snyder,
Professor J. W. Young.

Members of the Council to Serve until December, 1919,

Professor G. C. Evans, Professor L. A. Howland,
Professor G. H. Ling, Professor R. L. Moore.

The following papers were read at the annual meeting:

(1) Professor J. E. Rowe: "The relation of singularities of the rational quintic in space to loci of the rational plane quintic."

(2) Dr. C. A. Fischer: "Linear functionals of n-spreads."

(3) Professor H. B. Mitchell: "Geometrical limits for the imaginary roots of a polynomial with real coefficients."

(4) Professor Arnold Emch: "A theorem on the curves described by a spherical pendulum."

(5) Mr. J. K. Whittemore: "Spiral minimal surfaces."

(6) Dr. J. R. Kline: "Concerning the complement of a countable infinities of point sets of a certain type."

(7) Professor L. L. Dines: "On projective transformations in function space."

(8) Professor C. C. Grove: "Foundation of the correlation coefficient."

(9) Professor O. E. Glenn: "Preliminary report on invariant systems belonging to domains."

(10) Dr. Norbert Wiener: "Certain formal invariances in Boolean algebras."

(11) Professor L. P. Eisenhart: "Theory of transformations T of conjugate systems."

(12) Professor E. V. Huntington: "Complete existential theory of the postulates for serial order."

(13) Professor E. V. Huntington: "Complete existential theory of the postulates for well ordered sets."

(14) Professor Daniel Buchanan: "Orbits asymptotic to an isosceles-triangle solution of the problem of three bodies."

(15) Professor Daniel Buchanan: "Asymptotic satellites about the straight-line equilibrium points."

(16) Professor Daniel Buchanan: "Asymptotic satellites about the equilateral-triangle equilibrium points."

(17) Dr. G. M. Green: "Isothermal nets on a curved surface."

(18) Dr. A. L. Miller: "Systems of pencils of lines in ordinary space."

(19) Dr. W. L. Hart: "On an infinite system of ordinary differential equations."

(20) Dr. W. L. Hart: "Linear differential equations in infinitely many variables."

(21) Professor E. V. Huntington: "A set of independent postulates for cyclic order."

(22) Professor E. V. HUNTINGTON: "Sets of independent postulates for order on a closed line."

(23) Professor FRANK MORLEY: "The cubic seven-point and the Lüroth quartic."

(24) Professor J. L. COOLIDGE: "The intersections of a straight line and hyperquadric."

(25) Professor A. D. PITCHER: "Biextremal connected sets."

(26) Professor H. H. MITCHELL: "Proof that certain ideals in a cyclotomic realm are principal ideals."

(27) Professor H. H. MITCHELL: "On the asymptotic value of sums of power residues."

(28) Professor EDWARD KASNER: "Certain systems of curves connected with the theory of heat."

(29) Miss TERESA COHEN: "On a concomitant curve of the planar quartic."

(30) Professor P. F. SMITH: "A theorem for space analogous to Cesàro's theorem for plane isogonal systems."

(31) Professor W. E. STORY: "Some variable three-term scales of relation."

(32) Professor E. W. BROWN: presidential address: "The relations of mathematics to the natural sciences."

(33) Professor A. O. LEUSCHNER, vice-presidential address, Section A, American association for the advancement of science: "Derivation of orbits—theory and practice."

(34) Professor G. D. BIRKHOFF: "A class of series allied to Fourier series."

(35) Professor G. D. BIRKHOFF: "Note on linear difference equations."

(36) Professor G. A. MILLER: "Groups generated by two operators of the same prime order such that the conjugates of the one under the powers of the other are commutative."

(37) Mr. H. S. VANDIVER: "On the power characters of units in a cyclotomic field."

(38) Professor HENRY TABER: "On the structure of finite continuous groups."

Dr. Wiener was introduced by Professor Huntington, Miss Cohen by Professor Morley. Professor Leuschner's vice-presidential address was read by Professor Haskell. The papers of Professor Rowe, Professor Emch, Professor Dines, the first two papers of Professor Buchanan, Dr. Hart's first paper, and the papers of Professor Story and Professor Taber were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In a previous paper (*Transactions,* volume 13, No. 3, pages 395–404) Professor Rowe has pointed out how co-combinant curves of the rational plane quintic R_2^5, whose parametric equations are written *with* binomial coefficients, may be transformed into the parameters of singularities on the rational space quintic R_3^5, whose parametric equations are written *without* binomial coefficients. For instance, it was shown there that the point projection of a point of the plane upon the R_2^5 transforms into the parameters of the eight tetratactic planes of the R_3^5.

The present paper is a completion of part of the work suggested in the paper cited above. The simplest singularities of a rational plane curve are flex tangents, double tangents, and double points. The simplest singularities of a rational space curve are tetratactic planes, touching-osculating planes, tritangent planes, unisecant-tangent lines (i. e., lines tangent to the curve at one point and meeting it in one other point), and quadrisecant lines. Obviously the R_3^5 does not have any tritangent planes, but it has all of the other singularities. The author has shown how to derive the parameters of these from the proper covariant loci of the R_2^5.

The new results obtained may be stated briefly as follows: (1) The quartic (20), page 399 of the paper referred to above, transforms into a quartic whose roots are the parameters of the points cut out of the R_3^5 by its unique quadrisecant line. (2) Through every point of the plane pass twelve osculant conics of the R_2^5; the twelvic yielding the parameters of these transforms into the twelvic whose roots are the parameters of the twelve touching-osculating planes of the R_3^5. (3) Through every point of the plane pass twelve flex tangents of twelve osculant cubics of the R_2^5; the twelvic whose roots are the parameters of these cubics transforms into the twelvic whose roots are the parameters of the twelve unisecant-tangent lines of the R_3^5.

2. It has been proved by Riesz that a linear functional of a curve which is continuous with zeroth order can be expressed as a Stieltjes integral. The multiple Stieltjes integral has been defined by Fréchet, and proved to be a linear functional.

In the present paper Dr. Fischer has found some additional properties of such integrals, and proved that any linear functional of an n-spread which is continuous with zeroth order can be expressed as a multiple Stieltjes integral.

3. At the October meeting of the Society, Professor Mitchell presented a number of criteria whereby geometrical limits for the position of a pair of imaginary roots of an algebraic polynomial with real coefficients might be determined by inspection from the relative position of its real roots and bend points. In the present paper the method is extended so as to yield precise geometrical constructions when the polynomial has but two imaginary roots, and, in the case of more than two, to isolate the separate roots geometrically within regions that can be made small at will.

4. Professor Emch's paper appeared in full in the February BULLETIN.

5. It is well known that corresponding points of a family of associated minimal surfaces lie on an ellipse. In another paper Mr. Whittemore has found the locus of the vertices of these ellipses for any such family, and shown that, if the minimal surfaces are applicable to a surface of revolution, the locus consists of two parts, a plane and a surface of revolution. In the present paper it is proved that the only other minimal surfaces for which part of the locus is a plane are the spiral minimal surfaces, and that in this case the rest of the locus is a certain spiral surface. The author has then made a study of spiral minimal surfaces and proved certain properties of these surfaces, in many cases analogous to properties of minimal surfaces applicable to surfaces of revolution. These properties are related to the spiral curves of the surfaces, and concern lines of curvature, asymptotic lines, double lines, evolute surfaces, etc.

6. Dr. Kline's paper appears in full in the present number of the BULLETIN.

7. The transformations considered by Professor Dines are of the form

$$f_1(x) = \frac{a(x) + b(x)f(x) + \int_0^1 c(x, y)f(y)dy}{d + \int_0^1 e(y)f(y)dy}, \tag{1}$$

where the functions a, b, c, d, e satisfy the conditions:

1. All are continuous for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$;
2. The absolute value of $b(x)$ is greater than a positive constant on the interval $0 \leqq x \leqq 1$;
3. The equation $a(x) + b(x)f(x) + \int_0^1 c(x, y)f(y)dy = 0$ has a unique continuous solution $f(x) = h(x)$;
4. $d + \int_0^1 e(y)h(y) \neq 0$.

The set of all transformations of this type form a group under which the lines of function space are interchanged among themselves. Properties analogous to the properties of the corresponding projective transformations in space of n dimensions are obtained. Among other things it is shown that every finite transformation generated by the general infinitesimal projective transformation in function space, defined by Kowalewski, can be expressed in the form (1).

8. An analysis of the work of educational psychologists along the line of transfer and of the mathematics they have employed has led to the careful study of the foundations of the correlation coefficient, and cognate statistical measures discussed in this paper by Professor Grove.

9. Professor Glenn shows that, for a binary form, under general binary transformations, there exist complete systems of concomitants belonging to each of several domains of rationality. The methods by which the systems for these domains are constructed in case of the general transformation apply also to concomitants of substitutions constituting a special subset of the general transformations.

10. Dr. Wiener proves that the only operations in a Boolean algebra which possess formal properties such as logical addition and multiplication must possess are derivable from these by a one-one transformation belonging to the algebra. It is shown that negation remains invariant under all such transformations, and hence occupies a unique formal position in the algebra. The conditions under which an operation possesses such a unique formal position are discussed. The paper appeared in full in the January number of the *Transactions*.

11. When there is a one-to-one correspondence between the points of two surfaces S and S_1 of such a sort that the developables of the congruence of lines joining corresponding points meet S and S_1 in a conjugate system of curves on these surfaces, then S and S_1 are said to be in the relation of a transformation T. In several former papers Professor Eisenhart discussed certain types of these transformations, in particular those for which the conjugate systems on the two surfaces had their point invariants or their tangential invariants equal. In the present paper he develops the general theory of transformations T of a surface defined in either point or tangential coordinates. The well-known transformations D_m of isothermic surfaces are transformations T. The general transformations T admit a theorem of permutability reducing for the case of transformations D_m to the theorem due to Bianchi for the latter. The paper appeared in full in the January number of the *Transactions*.

12–13. Professor Huntington's two papers appear in full in the present number of the BULLETIN.

14. If two finite spheres of equal masses revolve about their common center of gravity, then an infinitesimal body subject to their attraction will describe a periodic orbit, under suitable initial conditions. This orbit is linear, being in the straight line through the center of gravity of the finite bodies and perpendicular to the plane of their motion. In a former paper (*Proceedings of the London Mathematical Society*, July, 1915), Professor Buchanan discussed the periodic oscillations about this linear orbit when the infinitesimal body is given an initial displacement from it. In the present paper, orbits are determined which are asymptotic to this linear orbit. The solutions are expansible as power series in a parameter which may denote the initial distance of the infinitesimal body from the linear orbit.

15. In this paper Professor Buchanan discusses orbits which are asymptotic to the periodic orbits of Class A and of Class B of the oscillating satellites about the straight-line equilibrium points as determined by Moulton in his Periodic Orbits, Chapter V.

16. In his third paper, Professor Buchanan discusses the

asymptotic satellites about the Lagrangian equilateral-triangle equilibrium points of the problem of three bodies. In Chapter IX of Moulton's Periodic Orbits, Buck determined two and three-dimensional periodic orbits about each equilibrium point. The two-dimensional periodic orbits were shown to exist only when one of the finite bodies is relatively small with respect to the other, but this restriction is not necessary for the existence of the three-dimensional orbits. In the present paper, however, it is shown that both the two and three-dimensional asymptotic orbits exist only when the finite bodies are more nearly equal than in the case of the two-dimensional periodic orbits. The two-dimensional asymptotic orbits approach the equilibrium points and not the corresponding periodic orbits. The three-dimensional asymptotic orbits, however, approach the three-dimensional periodic orbits.

17. An isothermal net on a curved surface is usually described as a net which divides the surface into infinitesimal squares. This of course is not properly speaking a geometric characterization of isothermal nets. Dr. Green gives two purely geometric characterizations of isothermal nets, one of which is easy to prove but requires a lengthy description; the other, however, though not so easy to prove, may be briefly described as follows. In previous papers Dr. Green has defined and made use of a certain geometric relation R, by means of which to a line l passing through any point of a curved surface is made to correspond a unique line l' in the corresponding tangent plane, the line l' being determined entirely by the line l and the parametric net of curves on the surface. Suppose the parametric net is any orthogonal net, and the line l is a normal to the surface. Then a unique line l' is determined in the corresponding tangent plane, so that corresponding to the congruence of normals l of the surface one obtains a congruence of lines l'. The developables of the congruence of lines l' correspond to a conjugate net on the surface, if and only if the parametric net is isothermal.

This obviously affords also an elegant characterization of isothermic surfaces, i. e., surfaces whose lines of curvature form an isothermal net. The other characterization of isothermal nets breaks down in this case; it also involves a use of the relation R, and the proof is so direct that the theorem is interesting in spite of its failure when the lines of curvature are parametric.

18. The paper of Dr. Miller, in its three parts, treats of the projective differential properties of systems of ∞^1, ∞^2, and ∞^3 pencils of lines in ordinary space by means of the synthetic methods of Segre.

In part II the congruence determined by two nearby pencils of the system is found to be more interesting than, although closely allied to, the tangent linear congruence. The locus of the congruences determined by a pencil and all the infinitely near pencils of the system is found to be a tetrahedral complex of the type [(22) (11)]. Two interesting "developable" cases of these systems are discussed, one in which all the tangent linear congruences to the system at lines of a pencil have in common a pair of pencils with a common line, and one in which all these tangent congruences coincide. The locus of the centers of the pencils of the system and the envelope of their planes are found to be the focal surfaces of a director congruence to which all the pencils of the system are tangent in the first developable case, while in the second they coincide.

In part III the systems of ∞^3 pencils are classified according to the various kinds of two-parameter families of which they are the locus. This leads to a brief projective differential study of Pfaffian differential equations.

19. In a previous paper* Dr. Hart established the unique existence of a solution of the infinite system of ordinary differential equations

$$(1) \qquad \frac{dx_i}{dt} = f_i(x_1, x_2, \cdots; t) \qquad [x_i(t_0) = a_i; i = 1, 2, \cdots],$$

in which the variables were real and satisfied

$$(2) \qquad |t - t_0| \leqq b, \qquad |x_k - a_k| \leqq r_k < r.$$

In the former paper the results were obtained, by means of a generalization of the Picard approximation process, under the assumption that the f_i satisfied a continuity condition and, also, a condition with respect to the variables $(x_1, x_2, \cdots)$

* *Proc. National Academy of Sciences*, vol. 2 (1916), p. 309.

analogous to the classical Lipschitz condition. In the present paper there is established

Theorem I. Let the functions f_i be completely continuous in the region (2). Suppose that the maxima M_i (which exist) of the $|f_i(x_1, x_2, \cdots; t)|$ satisfy $M_i \leqq r_i M$. Then there exists *at least* one solution $\xi(t) = [x_1(t), x_2(t), \cdots]$ of (1), $x_i(t_0) = a_i$, which is defined and continuous for $t_0 \leqq t \leqq c$ where c is the smaller of b and $1/M$.

The solution is obtained by a generalization of the Cauchy polygon method; on adding the Lipschitz condition of the previous paper it is readily established that two solutions $\xi(t)$ cannot exist.

In the present paper Dr. Hart also treats a system of differential equations in general analysis. Among the particular instances of this general theory are found the case of a system of n differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n; t) \quad (i = 1, 2, \cdots, n),$$

where the f_i need not satisfy a Lipschitz condition, and also the case of system (1) under the hypotheses of Theorem I.

20. In considering finite systems of ordinary differential equations it is found that linear systems, in particular, have a great number of interesting properties. Many of these are connected with the notion of fundamental sets of solutions. Dr. Hart, in his second paper, considers a class of infinite systems of linear differential equations of the form

$$(1) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{\infty} k_{ij}(t)x_j + g_i(t) \quad (i = 1, 2, \cdots; |t| \leqq r),$$

in which $(x_1, x_2, \cdots)$ are complex values for which $\sum_{i=1}^{\infty} |x_i|^2$ converges. The infinite matrix $(k_{ij}(t))_{i,\, j=1,\, 2,\, \ldots}$ is assumed to be *limited* in the sense used for the term by Hilbert. The $(g_1(t), g_2(t), \cdots)$ are such that $\sum_{i=1}^{\infty} |g_i(t)|^2$ converges for every t. The elements k_{ij} and g_i are assumed to be analytic in t and are regular for $|t| \leqq r$. For the system (1) an infinite matrix of solutions $A(t)$ is said to constitute a fundamental set of solutions in case A is limited and possesses a unique limited reciprocal. The system (1), in relation to this notion of fundamental sets of solutions, is found to have many of the familiar properties of finite systems of linear equations.

21. Professor Huntington shows that a class K is arranged in cyclic order (that is, in circular order with a definite sense around the circle) by a triadic relation $R(ABC)$, if the following five postulates are satisfied: (A) If ABC, then BCA. (B) If A, B, C are distinct, then at least one of the six triads, ABC, ACB, etc., is true. (C) If ABC is true, then ACB is false. (D) If ABC is true, then A, B, C are distinct. (1) If A, B, X, Y are distinct, and XAB and AYB, then XAY. The independence of these five postulates is established, and the relations between these postulates and the postulates for betweenness (BULLETIN, volume 23, pages 70–71, November, 1916) is discussed. A fuller abstract of the present paper may be found in the *Proceedings* of the National Academy of Sciences, volume 2, pages 630–631, November, 1916.

22. The theory of order on a closed line without distinction of sense is the same as the theory of the separation of pairs of points (begun by Vailati in 1895), and is based on a tetradic relation $ABCD$. Professor Huntington starts with the following basic list of fourteen postulates: (A_1) If $ABCD$, then $BCDA$. (A_2) If $ABCD$, then $DCBA$. (B) If A, B, C, D are distinct, then at least one of the twenty-four possible permutations $ABCD$, $ABDC$, etc., is a true tetrad. (C) If $ABCD$ is true, then $ABDC$ is false. (D) If $ABCD$ is true, then A, B, C, D are distinct. If A, B, C, X, Y are distinct, and $ABXC$ and $ABCY$, then: (1) $ABXY$; (2) $BXCY$; (3) $AXCY$. If A, B, C, X, Y are distinct, and $ABCX$ and $ABCY$, then: (4) $ABXY$ or $ABYX$; (5) $ACXY$ or $ACYX$; (6) $BCXY$ or $BCYX$; (7) $ABXY$ or $ACYX$; (8) $ABXY$ or $BCYX$; (9) $ACXY$ or $BCYX$. From this basic list, which contains many redundancies, eight different sets of independent postulates are obtained as follows (postulates A_1, A_2, B, C, D being required in each set): (I) 1, 4; (II) 1, 5; (III) 1, 6; (IV) 1, 7; (V) 1, 8; (VI) 1, 9; (VII) 2; (VIII) 3. Of these eight sets, the seventh appears to be the only one previously known.

23. Professor Morley's paper is in abstract as follows: Aronhold's construction of a curve of class four from seven

given points a_i gives a quartic of Lüroth's type when the seven points have each the same polar line as to a conic and a cubic (Bateman, *American Journal*, volume 36, 1914). If so, the nodal tangents of a cubic with a double point a_i and on the other six points will be apolar to the conic. Therefore a skew symmetric form $a^5x^2y^2$, which when y is a_i is the nodal tangents, will be such that the 6-rowed determinant of its coefficients vanishes. This determinant is the square of a Pfaffian of degree 15 in a_i, and further the Pfaffian vanishes when six of the points are on a conic. There is then a *cubic* invariant of 7 points which vanishes when they are an Aronhold set of a Lüroth quartic. This is the only cubic invariant of seven points. For six given points, there is then a cubic curve, on the six. Mapping on a cubic surface we have 36 plane sections (one for each double six), the locus of points on the surface from which the tangent cones are of Lüroth's type. Hence the degree of Lüroth's invariant is 54.

24. Professor Coolidge's paper contains a symmetrical form for the solution of $n - 1$ linear and one quadratic equation in $n + 1$ homogeneous variables.

25. In his thesis Fréchet considers the theory of functions μ of a variable q which ranges over a general class Q on which is defined a distance function δ conditioned to meet the needs of the theory. Professor Pitcher considers such systems $(Q; \delta)$ where δ is such that $\delta(qq) = 0$ and $\delta(q_1q_2) = \delta(q_2q_1)$. In particular he gives a set of three very simple conditions on the system $(Q; \delta)$ which are completely independent and which are both necessary and sufficient for a so-called uniformly proper theory of continuous functions on the set Q, where the term, uniformly proper, is used in a sense likely to be admitted by anyone who gives the matter careful consideration.

26. If l is a prime of the form $4n + 3$, and p a prime of the form $ml + 1$, Jacobi and Cauchy have shown that a certain power of a prime ideal factor of p in the quadratic realm $k(\sqrt{-l})$ is always a principal ideal. Dirichlet later showed that the exponent is the number of classes of ideals in the realm. The result first mentioned has been extended by Eisenstein and Stickelberger to primes not of the form $ml + 1$. Professor Mitchell obtains a more general result by

considering a cyclotomic realm of degree $2m$, determined by $2m$ periods, each containing n lth roots of unity, where n is odd, and $l = 2mn + 1$. If p is any prime that in this realm is the product of $2m$ conjugate prime ideal factors, he shows that a certain power of the product of m of these factors, if properly chosen, is a principal ideal. The exponent in this case is identical with the number given by Kummer as the so-called first factor of the class number.

27. If l is a prime of the form $4n + 3$, Dirichlet has shown that the sum of the quadratic residues of l that lie between 0 and l is always less than the sum of the non-residues in the same interval. Certain limits between which these sums must lie have been given by Stern, and somewhat similar limits may readily be deduced from some results obtained by Holden. Professor Mitchell obtains much closer limits for these sums, by means of which he shows that their ratio approaches the limit 1, as l increases indefinitely. Similar results are obtained for higher residues and a generalization is given for composite moduli.

28. The usual theory of isothermal systems of curves is based on the Laplace equation $\varphi_{xx} + \varphi_{yy} = 0$, derived on the assumption that the heat is in equilibrium. Professor Kasner studies more general systems of curves (for example the curves of constant temperature in a weather map) where this assumption is not valid. In particular, the cases where the rate of change of the temperature with respect to the time is independent either of the time or of the position are investigated in detail. In the first case we are led to a certain type of linear doubly infinite systems of curves. In the second case we have, as in the case of equilibrium, merely a simply infinite system $\varphi(x, y) =$ constant, but the fundamental partial differential equation is $\varphi_{xx} + \varphi_{yy} = 1$.

29. Miss Cohen's paper is in abstract as follows: The conic determined by a line ξ and the four tangents at its intersections with a quartic curve is of degree five in the coefficients of the quartic and eight in ξ. By use of special cases it can be found in terms of known comitants of the quartic. The conic breaks up when ξ is such that three of the tangents are on a point; therefore the discriminant of the conic is the locus of such

lines. The singular lines of this locus are: (1) the twenty-four stationary lines of the quartic, which are double lines with one contact, the flex; (2) the twenty-eight double lines of the quartic, which are double lines with the same contacts as with the quartic; (3) the twenty-one lines for which four tangents meet on a point, which are quadruple lines. The first two of these account for all the common lines of the locus and the quartic.

30. Cesàro's theorem for an isogonal system of plane curves is to the effect that the osculating circles of the curves through a given point pass through a second point. A companion theorem for an equitangential system has been established by Scheffers (*Mathematische Annalen,* volume 60 (1905), page 491), namely, the osculating circles of the curves of such a system which touch a given line touch also a second line. In Professor Smith's paper it is shown that simple transformations of the surface elements satisfying a partial differential equation of the first order lead, in one case, to an isogonal system of surface bands, in the other to an equitangential system of these bands. For each point of a given surface band two circles exist, each in a plane normal to the band at the point, such that the first circle osculates all surfaces containing the band, and the second osculates the developable surface upon which the band lies. Consideration of the circles of the first class for an isogonal system of surface bands leads to a theorem analogous to Cesàro's, while for an equitangential system of bands the circles of the second class play the rôle of the osculating circles in Scheffer's theorem.

31. Professor Story gives a proof of the theorem: an infinite succession of rational or irrational real numbers y_1, y_2, y_3, y_4, $\cdots$ of which each is expressible in terms of the preceding three by an equation of the form

$$y_h = L_h y_{h-1} + M_h y_{h-2} + N_h y_{h-3} \quad \text{for } 4 \leqq h$$

converges to a definite finite limit if the first three y's are finite and the coefficients L_h, M_h, N_h, for each integral suffix h as great as 4 are real numbers that satisfy the conditions

$$L_h + M_h + N_h = 1, \quad 0 < N_h < L_h, \quad 0 \leqq M_h, \quad \alpha < L_h,$$

where α is an a priori given positive proper fraction that is appreciably greater than 0. The convergence of a regular continued fraction of the second order (in the cases in which such a fraction is usable) follows from the special case of this theorem in which $\alpha = \frac{1}{3}$.

32. President Brown's address appeared in full in the February BULLETIN.

34. The most notable properties of the series of orthogonal or biorthogonal functions defined by linear differential equations of the second order and linear boundary conditions are: (1) the functions satisfy these boundary conditions, (2) they have a certain asymptotic form, and (3) the series may be used to represent wide classes of arbitrary functions.

The functions appearing in the series are also given as the solutions of a homogeneous linear integral equation of Fredholm type with the corresponding Green's function $G(x, y)$ for kernel.

Professor Birkhoff shows that, if $H(x, y)$ be any continuous kernel satisfying the given boundary conditions and possessing a discontinuous first derivative for $x = y$ like the Green's function, the solutions of the integral equation in H will form an orthogonal or biorthogonal series with properties (1), (2), (3), at least if simple further conditions be imposed. A converse statement is also established.

35. In the theory of linear difference equations two fundamental sets of solutions have been defined by means of their asymptotic properties in a half plane bounded by a straight line parallel to the axis of imaginaries and lying in the complex plane of the independent variables. Professor Birkhoff shows that similar sets of solutions exist for half planes bounded by any line not parallel to the axis of reals, and that these solutions may be made to take the place of the fundamental sets referred to.

36. Professor Miller's paper appears in full in the present number of the BULLETIN.

37. If p is a prime integer, $\alpha = e^{2i\pi/p}$, and $\mathfrak{p}$ is an ideal prime in the algebraic field Ω defined by α, then

$$\omega^{[N(\mathfrak{p})-1]/p} \equiv \alpha^a \pmod{\mathfrak{p}},$$

where a is some integer less than p, ω is an integer in Ω, and $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$. Then we write

$$\alpha^a = \{\omega/\mathfrak{p}\}.$$

If Ω is a *regular* field and ω is a unit therein, then an explicit expression was given by Kummer for the exponent a. Let

$$\{\omega/\mathfrak{m}\} = \{\omega/\mathfrak{p}\}\{\omega/\mathfrak{q}\}\{\omega/\mathfrak{r}\} \cdots \quad (\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{r} \cdots),$$

where $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\cdots$, are prime ideals in any field Ω. Mr. Vandiver obtains in the present paper an explicit expression for $\{\omega/\mathfrak{m}\}$, where $\mathfrak{m}$ is a principal ideal in *any* cyclotomic field defined by a pth root of unity, and ω is included among certain types of units in the field.

38. Let $X_1, \cdots, X_r$ be the infinitesimal transformations of a finite continuous group with no exceptional infinitesimal transformation; and let $A_1, \cdots A_r$ be the matrices of any system whatever of independent infinitesimal linear homogeneous transformations generating a group with the same structure as G. The matrices $E_1, \cdots, E_r$ of the infinitesimal transformations of the group Γ adjoint to G constitute such a system. Denoting by SM the sum of the constituents in the principal diagonal of any matrix M, Professor Taber finds that $S(\Sigma\alpha_iA_i)^m$, for any positive integer m, is an invariant of the adjoint group Γ, and thus the coefficients of the characteristic equation of the general infinitesimal transformation $\Sigma\alpha_iA_i$ of the group $A_1, \cdots, A_r$ are invariants of Γ. Among these invariants one is of especial importance, namely, the quadratic form $S(\Sigma\alpha_iA_i)^2$. If the discriminant of this form is zero, and of nullity p, there are just p independent solutions of the equations $S(\Sigma\alpha_iA_i)A_k = 0$ $(k = 1, 2, \cdots, r)$; and if $\Sigma\alpha_i^{(q)}A_i$ for $q = 1, 2, \cdots, p$, are any p independent solutions of these equations, then $\Sigma\alpha_i^{(q)}X_i$, for $q = 1, 2, \cdots, p$, generate an invariant subgroup of G with p parameters. On the other hand, if the discriminant is not zero, the group G is perfect; and is either simple or semi-simple, unless possibly G contains an invariant subgroup every infinitesimal transformation of which is exceptional. For the case in which $A_1, \cdots, A_r$ are respectively identical with $E_1, \cdots, E_r$, these theorems are known. When the discriminant is not zero, the adjoint group is orthogonal, or becomes so by a suitable choice of the infinitesimal transformations of G; and in this case, if c_{ijk}

for $i, j, k = 1, 2, \cdots, r$, are the structural constants of G, we have $c_{ijk} + c_{ikj} = 0$.

F. N. COLE,
Secretary.

COMPLETE EXISTENTIAL THEORY OF THE POSTULATES FOR SERIAL ORDER.

BY PROFESSOR EDWARD V. HUNTINGTON.

(Read before the American Mathematical Society, December 27, 1916.)

THE purpose of this note is to establish the "complete independence"—in the sense defined by E. H. Moore*—of each of three different sets of postulates for serial order.

The first set of postulates (set A) is new and will be found more convenient for many purposes than either of the other sets. Set B dates back to Vailati, 1892.† Set C (a modification of set B and now widely used) was introduced by the present writer in 1905.‡

The universe of discourse considered in each of these sets is the universe of all systems (K, R), in which K is a class of elements, $A, B, C, \cdots$, and R is a dyadic relation; the notation $R(AB)$, or briefly AB, meaning that the relation R holds

* E. H. Moore, "Introduction to a form of general analysis," Yale University Press (1910), p. 82. An interesting example of a proof of complete independence is given by R. D. Beetle, "On the complete independence of Schimmack's postulates for the arithmetic mean," *Math. Annalen*, vol. 76 (1915), pp. 444–446. [Compare R. Schimmack, "Der Satz vom arithmetischen Mittel in axiomatischer Begründung," *Math. Annalen*, vol. 68 (1909), pp. 125–132.] For a similar discussion of an *almost* completely independent set of postulates, see L. L. Dines, "Complete existential theory of Sheffer's postulates for Boolean algebras," this BULLETIN, vol. 21 (1915), pp. 183–188. [Compare H. M. Sheffer, "A set of five independent postulates for Boolean algebras, with application to logical constants," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 481–488.]

† G. Vailati, "Sui principî fondamentali della geometria della retta," *Rivista di Matematica*, vol. 2 (1892), pp. 71–75; B. Russell, Principles of Mathematics, vol. 1 (1903), pp. 203, 218–219.

‡ E. V. Huntington, "The continuum as a type of order," reprinted from the *Annals of Mathematics*, vols. 6 and 7 (1905), especially vol. 6, pp. 157–158; second edition, Harvard University Press, 1917, pp. 10–11, J. W. Young, Fundamental Concepts of Algebra and Geometry (1911), p. 68; A. N. Whitehead and B. Russell, Principia Mathematica, vol. 2 (1912), p. 513. (In the present terminology of Whitehead and Russell, a relation which satisfies postulate 1 is said to be "contained in diversity.").