THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and eighty-ninth regular meeting of the Society was held in New York City on Saturday, February 24. The morning session sufficed for the presentation of the brief list of papers. The attendance included the following twenty-six members:

Mr. D. R. Belcher, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor C. C. Grove, Mr. S. A. Joffe, Professor Edward Kasner, Dr. J. R. Kline, Mr. Harry Langman, Professor R. L. Moore, Mr. G. W. Mullins, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor W. B. Stone, Mr. H. E. Webb, Professor H. S. White, Mr. J. K. Whittemore.

Professor H. S. White occupied the chair, being relieved by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Professor H. P. Kean, McHenry College; Mr. Ralph Keffer, Harvard University; Mr. H. C. M. Morse, Harvard University; Dr. F. D. Murnaghan, Rice Institute; Mr. G. E. Raynor, University of Washington; Dr. S. P. Shugert, University of Pennsylvania; Mr. G. W. Smith, University of Illinois; Mr. J. S. Taylor, University of California; Dr. L. E. Wear, University of Washington; Dr. H. N. Wright, University of California. Four applications for membership in the Society were received.

It was decided to hold the next summer meeting of the Society at Cleveland, Ohio, on September 4–5. The Mathematical Association of America will meet at Cleveland on September 6–7. The two organizations have appointed a joint committee on arrangements for the meetings, consisting of Professors Focke, Huntington, Pitcher, D. T. Wilson, and Secretaries Cole and Cairns.

At the annual meeting of the Society, the Council placed itself on record as desiring to cooperate with the National Research Council in forwarding the interests of research. At the February meeting, a committee was appointed to confer with the chairman of the Mathematics Committee of the Research Council, Professor E. H. Moore, in regard to the selection of the members of that committee.

By the will of the late Professor L. L. Conant, who was a member of the Society from 1892 to 1916, the sum of $10,000 is left to the Society, subject to Mrs. Conant's life interest. The will provides that the income of this bequest "shall be offered once in five years as a prize for original work in pure mathematics." This generous gift, a noble monument to the donor, should do much for the promotion of higher mathematical aims in this country. For many years the Society has consistently pursued these aims, with a success far outrunning what might have been expected from its modest financial resources. With greater means in the way of general or special funds, it could accomplish still more. To anyone who is able to give for science, the Society presents itself as an experienced and beneficent administrator.

The following papers were read at the February meeting:

(1) Dr. A. R. SCHWEITZER: "The iterative compositions of a function of $n + 1$ variables ($n = 1, 2, 3, \cdots$)."

(2) Dr. A. R. SCHWEITZER: "Functional equations based on iterative compositions."

(3) Dr. D. F. BARROW: "An application of Fourier's series to probability."

(4) Professor EDWARD KASNER: "Degenerate cases in the theory of conduction of heat."

(5) Professor J. E. ROWE: "The equation of a rational plane curve derived from its parametric equations (second paper)."

(6) Professor R. L. MOORE and Dr. J. R. KLINE: "The most general closed plane point set through which it is possible to pass a simple continuous plane arc."

(7) Professor H. S. WHITE: "New proof of a theorem of von Staudt and Hurwitz."

(8) Professor HENRY TABER: "On the structure of finite continuous groups."

In the absence of the authors, the papers of Dr. Schweitzer, Dr. Barrow, Professor Rowe, and Professor Taber were read by title. Abstracts of the papers follow below.

1. Dr. Schweitzer constructs an exhaustive table of iterative compositions of a function of $n + 1$ variables ($n = 1, 2, 3, \cdots$). A symbolic representation for such a composition

is used, and by introducing a simple convention the compositions are readily ordered for each value of n. The construction of the table is based on the general principle implied by the following operation: If the function $f(x_0, x_1, \cdots, x_n)$ has some one of its arguments, say x_0, replaced by the function f, then this f is denoted by $f(x_{00}, x_{01}, \cdots, x_{0n})$ and the resulting composition is denoted by the symbol

$$[01, \cdots, n; (00, 01, \cdots, 0n), 1, \cdots, n].$$

More compactly, the author puts, e. g., $F(01) \equiv f(x_0, x_1)$, $F(01; 1) \equiv [F(01); 0, (10, 11)]$, etc. Similarly for three variables one obtains $F(012)$, $F(012; 1)$, etc., $F(012; 11)$, etc. Any iterative composition of a function of $n + 1$ variables is represented, then, by the symbol $F(012 \cdots n; i_1, i_2, \cdots)$ where i_1, i_2, etc., are certain integers. To illustrate, for $n = 1$ the iterative compositions are $f(x_0, x_1)$, $f\{x_0, f(x_{10}, x_{11})\}$, $f\{f(x_{00}, x_{01}), x_1\}$, $f\{f(x_{00}, x_{01}), f(x_{10}, x_{11})\}$, $f\{x_0 f[x_{10}, f(x_{110}, x_{111})]\}$, etc.

2. In Dr. Schweitzer's second paper a systematic application is made of the preceding iterative compositions of a function of $n + 1$ variables ($n = 1, 2, 3, \cdots$) to the genesis of functional equations, by means of the inversion, elimination, or iteration of some or all of the involved variables (the processes being taken separately or in combination). The author obtains three categories of functional equations, viz., the inversive, the eliminative, and the iterative, which, however, are not necessarily mutually exclusive. It is found that certain eliminative functional equations, including certain of those called "quasi-transitive," are implied by certain purely inversive functional relations. An important class of inversive functional equations is generated by the following requirement: Given the iterative composition $F(012 \cdots n; i_1, i_2, \cdots)$ and a substitution group G on the variables involved (leaving some or none of them fixed); to find the function $f\{x_1, x_2, \cdots, x_{n+1}\}$ which satisfies the functional equations expressed by the formal invariance of the composition $F(012 \cdots n; i_1, i_2, \cdots)$ under the substitutions of the group G. On the other hand extensive new categories of eliminative functional equations are obtained. A few of the theorems proved are as follows:

I. If $f\{f(x_1, x_2), f(x_3, x_4)\}$ is formally invariant under the cyclic group on x_1, x_2, x_3, x_4 then $f(x_1, x_2) = x^{-1}\{cx(x_1) + cx(x_2)\}$ where c is an arbitrary constant.

II. This theorem is a direct generalization of theorem I and incidentally generalizes Hayashi's generalization of Abel's functional equations. Instead of the symmetric group the cyclic group is used.

III. If $f\{f(x_1, x_2), f(x_3, x_4)\}$ is formally invariant under the non-cyclic group of order four on x_1, x_2, x_3, x_4, and if $f(x, x) = \text{const.}, f\{f(x_1, x_2)c_1\} = \lambda f(x_1, x_2)$, then $f(x_1, x_2) = x^{-1}\{cx(x_1) - cx(x_2)\}$, where c is an arbitrary constant.

IV. If $f\{x, f(t_1, t_2, y), f(t_1, t_2, z)\} = f(x, y, z)$, then $f(x, y, z) = x^{-1}\{\vartheta(x) - x(y) + x(z)\}$, where $\vartheta(x)$ is arbitrary.

V. If $f\{f(x, y), f[x, f(x, \cdots, f\{x, z\})]\} = f(z, y)$, where in the latter equation the symbol f occurs $n + 3$ times $(n = 1, 2, 3, \cdots)$, then $f(x, y) = x^{-1}\{x(x) + wx(y)\}$, where w is a root of the equation $w^n + w^{n-1} + \cdots + w + 1 = 0$.

3. This paper is the outgrowth of an attempt on the part of Dr. Barrow to solve a problem proposed by Professor E. W. Brown. A solution of the problem and a new proof of certain well known properties of frequency curves are obtained as applications of the main theorem of the paper which is as follows:

Let two quantities N and M have frequencies $f(x)$ and $\varphi(x)$ respectively, and let $F(x)$ denote the frequency of $M + N$. If these three functions are identically zero outside some interval $(-C, C)$, and capable of development in Fourier's series within this interval, then, letting a_i and b_i, α_i and β_i, A_i and B_i denote the ith coefficients of the cosines and sines in these series, we have

$$A_i + B_i\sqrt{-1} = C(a_i + b_i\sqrt{-1})(\alpha_i + \beta_i\sqrt{-1}).$$

Equating the real and imaginary parts enables us to calculate the coefficients in the development of $F(x)$ from those in the developments of $f(x)$ and $\varphi(x)$.

4. The partial differential equation for the conduction of heat in a plane determines the temperature as a function of the position (x, y) and the time t. For each solution $\varphi(x, y, t)$ there will be, in general, ∞^2 isothermal curves $\varphi(x, y, t) = c$.

Professor Kasner finds all solutions for which the doubly infinite system degenerates into a simply infinite system. There are three types: In the first $\varphi = f(x, y)$, the temperature not entering; in the second, $\varphi = f(x, y) + t$; in the third, $\varphi = e^t f(x, y)$. The differential equations for these types are respectively $\Delta f = 0$, $\Delta f = 1$, $\Delta f = f$, where $\Delta f \equiv f_{xx} + f_{yy}$.

5. Professor Rowe's paper appears in full in the present number of the BULLETIN.

6. In order that it may be possible to pass a simple continuous arc through a given closed point set M it is sufficient that M should contain no connected subset consisting of more than one point. Cf. F. Riesz, *Comptes Rendus*, volume 141 (1905), page 650, A. Denjoy, ibid., volume 151 (1910), page 140, and L. Zoretti, ibid., volume 142 (1906), page 763. It is clear that this is not a necessary condition. Professor Moore and Dr. Kline have obtained the following theorem:

In order that it may be possible to pass at least one simple continuous arc through a given closed point set M it is necessary and sufficient that every closed connected subset of M should be either a single point or a simple continuous arc k no point of which, except its end points, is a limit point of $M - k$.

7. The theorem concerning two tetrahedra inscribed in a twisted cubic curve was announced by von Staudt in his Beiträge, and proved by Hurwitz thirty years later. Professor White points out that a simpler proof is found by projecting the curve, from all its points, into a plane. Then the theorem in question is derived easily from the well-known situation of two triangles inscribed to a conic, which is the Poncelet theorem for triangles.

8. In Professor Taber's paper $X_1, \cdots, X_r$ are the infinitesimal transformations of a finite continuous group G, in which case $(X_i, X_j) = \Sigma_1^r c_{ijk} X_k$ $(i, j = 1, 2, \cdots, r)$. Any infinitesimal transformation $\Sigma\alpha_i X_i$ of G can be represented by the point $\alpha = (\alpha_1, \cdots, \alpha_r)$ in the space of $r - 1$ dimensions upon which the group Γ adjoint to G operates. If now $f(\alpha) \equiv f(\alpha_1, \cdots, \alpha_r)$ is an invariant of Γ, it follows that the alternant of $\Sigma\alpha_i X_i$ and any other infinitesimal transformation of G is represented by

a point which lies in the polar $r-2$ flat of α qua $f(\alpha)=0$. Whence it follows that any $r-2$ flat $\Sigma u_i\alpha_i=0$ is invariant to each of the infinitesimal transformations of Γ represented by the poles of this flat qua $f(\alpha)=0$; and thus, if the poles of this flat do not all lie in an $r-2$ flat, it follows that the infinitesimal transformations of G represented by points in the $r-2$ flat $\Sigma u_i\alpha_i=0$ generate an invariant subgroup of G.

If the adjoint group Γ has a quadratic invariant of non-zero discriminant, for a proper choice of the X's, we shall have $c_{ijk}+c_{ikj}=0$ $(i, j, k=1, 2, \cdots, r)$. In this case the condition that $f(\alpha)$ shall be invariant to Γ is that

$$\Sigma\alpha_iE_i\left(\frac{\partial f(\alpha)}{\partial\alpha_1}, \cdots, \frac{\partial f(\alpha)}{\partial\alpha_r}\right)=0,$$

where E_i is the matrix whose constituent in the μth row and νth column is $c_{i\nu\mu}$ $(i, \mu, \nu=1, 2, \cdots, r)$; or, what is the same thing, is that

$$\left(\Sigma\alpha_iX_i, \Sigma\frac{\partial f(\alpha)}{\partial\alpha_i}X_i\right)=0$$

for all values of the α's. Whence it follows, if $f(\alpha)$ is a second invariant of Γ, that the infinitesimal transformation Σu_iX_i is commutative with every infinitesimal transformation of G represented by a pole, qua $f(\alpha)=0$, of the $r-2$ flat $\Sigma u_i\alpha_i=0$; and, if these poles do not all lie in any $r-2$ flat, it follows that Σu_iX_i is an exceptional infinitesimal transformation.

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Secretary.

THE EQUATION OF A RATIONAL PLANE CURVE DERIVED FROM ITS PARAMETRIC EQUATIONS (SECOND PAPER).

BY PROFESSOR J. E. ROWE.

(Read before the American Mathematical Society, February 24, 1917.)

As this is the second article on the same subject published by the author in the BULLETIN, it is desirable to inform the reader at once that the method of deriving the equation of a rational plane curve from its parametric equations described in this paper is published not merely because it is a new