THE MODULAR DIFFERENCE OF CLASSES.

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IN the general analysis of numerical functions the modular difference of two numbers is an extremely valuable concept. This paper discusses an analogous concept in the calculus of classes. These classes may be logical classes in general or may be point sets. The following notation* is used in which capital letters denote classes.

AB = logical product of A and B.

$A + B$ = logical sum of A and B.

$\bar{A}$ = complement of A or CA in Borel's notation.

$A < B$ means A is included in B.

$A > B$ means A includes B.

$A - B$ is defined only when $A > B$ and is then equal to $A\bar{B}$.

The modular difference of two numbers a, b satisfies the two conditions

(1) when $a \geqq b$, $|a - b| = a - b$,

(2) $|a - b|$ is symmetrical in a and b.

By analogy the modular difference of two classes A, B denoted by $|A - B|$ should satisfy

(1) when $A > B$, $|A - B| = A\bar{B} = A - B$,

(2) $|A - B|$ is symmetrical in A and B.

Definition.—The modular difference of two classes is the logical sum of the logical product of the first into the complement of the second and the logical product of the second into the complement of the first.

$$|A - B| = \bar{A}B + A\bar{B} \qquad \text{(Def.).}$$

The reader is to note that the modular difference is defined even in cases where the difference according to Borel is not defined. That is to say, the modular difference is a direct operation on the two classes, not an operation on their difference.

Using the definition above and the analysis of symbolic logic the following properties can be proved.

* Borel, Théorie des Fonctions des Variables réelles, p. 16. Baldwin, Dictionary of Philosophy and Psychology, article "Symbolic Logic." Whitehead and Russell, Principia Mathematica, chapter on "Calculus of Classes."

Properties.

1(1). $|A - B| = A\bar{B} + \bar{A}B = (A + B)(\bar{A} + \bar{B}) = |\bar{A} - \bar{B}|$.

1(2). $\overline{|A - B|} = AB + \bar{A}\bar{B} = |\bar{A} - B| = |A - \bar{B}|$.

1(3). If $|A - B| = 0$, $A = B$ and vice versa.

(In this, 0 may be regarded as a set or class with no elements.)

2(1). $||A - B| - C| = |A - |B - C||$.

2(1) cor. If $|A - B| = C$, $A = |B - C|$.

2(2). $||A - B| - |B - C|| = |A - C|$.

2(2) cor. If $|A - B| = |B - C|$, $A = C$ and vice versa.

2(3). $|A - B||B - C||C - A| = 0$.

2(4). $C|A - B| = |CA - CB|$.

3(1). $|(A + B) - C| = |A - C||B - C| + |A - B|\bar{C}$.

3(2). $|AB - C| = |A - C||B - C| + |A - B|C$.

3(3). $|A - C| + |B - C| = |(A + B) - C| + |A - B|C$.

3(4). $|A - C| + |B - C| = |AB - C| + |A - B|\bar{C}$.

4(1). $|A - B| < A + B$.

4(2). $|A - C| < |A - B| + |B - C|$.

4(3). $|A - C||B - C| < |(A + B) - C| < |A - C| + |B - C|$.

4(4). $|A - C||B - C| < |AB - C| < |A - C| + |B - C|$.

5. $(A + B + C + \cdots + N) - ABC\cdots N = |A - B| + |A - C| + \cdots + |A - N|$.

The standard type of equation in symbolic logic having a unique solution is of the form

$$X\bar{A} + \bar{X}A = B.$$

Using our notation, this becomes $|X - A| = B$ or by 2(1) cor. the solution is $X = |B - A| = B\bar{A} + \bar{B}A$.

The properties under 4 are modular properties which are immediate deductions from the equations under 1(1), 2(2) and 3. Property 5 will be useful to us in what follows.

Definition.—A sequence of classes $S_1, S_2, \cdots, S_n, \cdots$ is said to be a standard sequence, or S-sequence, if $S_n > S_{n+1}$ for all values of n and if $\lim S_n = 0$, that is, if each class includes the next and if there is no element common to all.

A sequence is said to be decreasing if each class includes the next and to be increasing if each class is included in the next.

In this paper inclusion denotes logical inclusion, that is to say, equality is also inclusion in the sense that if $A = B$, then $A > B$ and $B > A$.

Limit.—A sequence of classes $A_1, A_2, \cdots, A_n, \cdots$ is said to possess a limiting class A if an A can be found such that $| A - A_n | < S_n$ for all values of n, where S_n is the nth member of a standard sequence. We proceed to prove that this definition coincides in content with that given by Borel (see reference above).

Firstly, if the sequence is decreasing and A is its limit by Borel,

$$| A - A_n | = A_n - A < A_n - A$$

and $A_n - A$ forms an S-sequence. Secondly, if the sequence is increasing and A is its limit by Borel,

$$| A - A_n | = A - A_n < A - A_n$$

and $A - A_n$ forms an S-sequence. Thirdly, if in general the sequence possesses a Borel limit A,

$$
\begin{aligned}
&| (A_m + A_{m+1} + \cdots + A_{m+p}) - A_n | \\
&\quad < | A_n - A_m | + | A_n - A_{m+1} | + \cdots + | A_n - A_{m+p} | \\
&\hspace{22em} \text{[Prop. 4(3)]} \\
&\quad < | A_n - A_{n+1} | + | A_n - A_{n+2} | + \cdots + | A_n - A_{m+p} | \\
&\hspace{22em} \text{if } m > n \\
&\quad < (A_n + A_{n+1} + \cdots + A_{m+p}) - A_n A_{n+1} \cdots A_{m+p} \\
&\hspace{22em} \text{[Prop. 5]} \\
&\quad < (A_n + A_{n+1} + \cdots) - A_n A_{n+1} \cdots, \\
&| (A_m + A_{m+1} + \cdots) - A_n | \\
&\qquad\qquad < (A_n + A_{n+1} + \cdots) - A_n A_{n+1} \cdots \quad \text{if } m > n.
\end{aligned}
$$

But the right-hand side forms a decreasing sequence and has the limit 0 if the limit of sequence A_n exists according to Borel. Thus the right-hand side is an S_n of an S-sequence. $|(A_m + A_{m+1} + \cdots) - A_n | < S_n$ for all values of $m > n$. Hence in the limit as m increases indefinitely

$$| \text{Borel limit } A - A_n | < S_n.$$

Hence the sequence A_n has the Borel limit A, according to our definition, whenever it has a limit by Borel's definition. Again, suppose that it has a limit A by our definition

$$| A - A_n | < S_n$$

of some S sequence.

$$
\begin{aligned}
&| (A_n + A_{n+1} + \cdots + A_{n+p}) - A | \\
&\quad < | A - A_n | + | A - A_{n+1} | + \cdots + | A - A_{n+p} | \qquad \text{[Prop. 4(3)]} \\
&\quad < S_n + S_{n+1} + \cdots + S_{n+p} \\
&\quad < S_n.
\end{aligned}
$$

Taking the limit as p increases indefinitely, if

$$
B_n = A_n + A_{n+1} + \cdots
$$
$$
| B_n - A | < S_n.
$$

But B_n forms a decreasing sequence and has a limit B which is the upper limiting set of the sequence A_n. Hence

$$
| B - B_n | < S_n'
$$

of some S-sequence; in fact

$$
\begin{aligned}
S_n' &= B_n - B \\
| B - A | &< | B - B_n | + | B_n - A | \qquad \text{[Prop. 4(2)]} \\
&< S_n' + S_n
\end{aligned}
$$

for all values of n. But $| B - A |$ is independent of n, hence

$$
\begin{aligned}
| B - A | &= 0 \\
B &= A \qquad \text{[Prop. 1((3].}
\end{aligned}
$$

Thus the upper limiting set $= A$. Similarly it can be proved that the lower limiting set $= A$ and therefore the sequence A_n has a limit according to Borel and this limit is A, the same as the limit by our definition.

Property 6(1).—The necessary and sufficient condition that a sequence of sets or classes A_n has a limit A according to Borel is that $| A - A_n | < S_n$ of some standard sequence.

Property 6(2).—The necessary and sufficient condition that a sequence of sets or classes A_n has a limit is that

$$
| A_n - A_{n+p} | < S_n
$$

of some S-sequence, the same for all values of p. For if the sequence A_n possesses a limit A

$$
| A - A_n | < S_n, \quad | A - A_{n+p} | < S_{n+p},
$$
$$
\begin{aligned}
| A_n - A_{n+p} | &< | A - A_n | + | A - A_{n+p} | \\
&< S_n + S_{n+p} \\
&< S_n.
\end{aligned}
$$

If, on the other hand,

$$| A_n - A_{n+p} | < S_n$$

$$\begin{aligned}(A_n + A_{n+1} + \cdots + A_{n+p}) &- A_n A_{n+1} \cdots A_{n+p} \\ &= | A_n - A_{n+1} | + | A_n - A_{n+2} | + \cdots + | A_n - A_{n+p} | \\ &< S_n + S_n + \cdots + S_n \\ &< S_n.\end{aligned}$$

In the limit when p increases indefinitely

$$(A_n + A_{n+1} + \cdots) - A_n A_{n+1} \cdots < S_n.$$

The left member is a sequence decreasing with n and it must have the limit 0 for S_n has the limit 0. Consequently the sequence A_n has a limit according to Borel.

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ON THE PRINCIPAL UNITS OF AN ALGEBRAIC DOMAIN $k(\mathfrak{p}, \alpha)$.

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Introduction.

THE following paper is the result of an investigation of a problem connected with the representation of the algebraic numbers in the form $\pi^\alpha \omega^\beta e^\gamma$.*

Throughout the discussion I shall use the following notation. By p I mean a rational prime and by $\mathfrak{p}$ any prime divisor of p. f is the degree of $\mathfrak{p}$, i. e., $N(\mathfrak{p}) = p^f$ and $\mathfrak{p}^\sigma$ is the highest power of $\mathfrak{p}$ contained in p. By π I mean a prime number of the domain $k(\mathfrak{p}, \alpha)$, where α is an arbitrary algebraic number. The numbers of $k(\mathfrak{p}, \alpha)$ are then of the form $a_\rho \pi^\rho + a_{\rho+1} \pi^{\rho+1} + \cdots$. A number in which $\rho = 0$ and a_ρ is relatively prime to $\mathfrak{p}$ is called a unit and in particular if $a_\rho = 1$ it is called a principal unit.

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* Hensel, *Crelle's Journal*, vol. 145.