THE TWENTY-FIFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fifth summer meeting of the Society was held, by invitation, at Dartmouth College on Wednesday, Thursday and Friday, September 4–6, 1918, connecting with the meeting of the Mathematical Association, which began on Friday morning. The joint dinner of the two organizations, on Thursday evening, was attended by fifty-six members and friends, who were greeted by Dean Laycock in the name of the College. At the joint session on Friday morning Professor A. G. Webster gave an address on "Mathematics of warfare."

The college dormitories were opened for the accommodation of the visitors, and meals were served in the commons. Headquarters and general gathering place between the sessions was provided in College Hall, where an informal reception was held on Wednesday evening. A letter of welcome from Business Director Keyes tendered the hospitality of the College to the visiting societies. Excursions into the country about Hanover were arranged for the closing days of the meetings. At the joint session a vote of thanks was extended to the college authorities for their generous cooperation toward a successful occasion.

The meeting included an evening session on Wednesday and the usual morning and afternoon sessions on Thursday, as well as the joint session on Friday morning. The attendance included the following forty-six members of the Society:

Professor H. L. Agard, Professor R. C. Archibald, Professor R. D. Beetle, Professor G. D. Birkhoff, Professor Daniel Buchanan, Professor W. D. Cairns, Professor W. B. Carver, Professor F. N. Cole, Professor Julia T. Colpitts, Dr. Lennie P. Copeland, Professor Louise D. Cummings, Dr. Mary F. Curtis, Professor C. H. Currier, Professor E. L. Dodd, Mr. T. C. Frye, Professor A. S. Gale, Professor O. E. Glenn, Mr. B. F. Groat, Professor C. F. Gummer, Professor J. G. Hardy, Professor C. N. Haskins, Professor E. V. Huntington, Professor W. W. Johnson, Professor Florence P. Lewis, Professor Helen A. Merrill, Dr. A. L. Miller, Professor F. M. Morgan, Professor G. D. Olds, Professor F. W. Owens, Professor Anna H. Palmié, Professor A. D. Pitcher, Professor J. M. Poor, Mr.

L. H. Rice, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Dr. Josephine R. Roe, Professor Clara E. Smith, Professor Sarah E. Smith, Professor H. W. Tyler, Professor Oswald Veblen, Professor C. A. Waldo, Professor A. G. Webster, Mr. J. K. Whittemore, Professor C. B. Williams, Professor F. N. Willson, Professor J. W. Young.

Professor W. W. Johnson presided at the opening session on Wednesday evening, Professor H. W. Tyler at the sessions on Thursday, and Professor G. D. Birkhoff at the joint session on Friday morning. The Council announced the election of the following persons to membership in the Society: Professor A. L. Candy, University of Nebraska; Mr. J. R. Carson, American Telephone and Telegraph Company; Mr. R. S. Hoyt, American Telephone and Telegraph Company; Dr. K. W. Lamson, Columbia University; Professor A. S. Merrill, University of Montana; Mr. F. H. Murray, Harvard University; Mr. H. W. Nichols, Western Electric Company; Professor W. E. Patten, Government Institute of Technology, Shanghai, China. Nine applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Dr. L. B. Robinson: "A curious system of polynomials."

(2) Professor G. A. Miller: "Groups generated by two operators whose relative transforms are equal to each other."

(3) Professor P. J. Daniell: "Differentiation with respect to a function of limited variation."

(4) Mr. B. F. Groat: "Models and hydraulic similarity."

(5) Professor L. C. Mathewson: "On the groups of isomorphisms of a system of abelian groups of order p^m and type $(n, 1, 1, \ldots, 1)$."

(6) Mr. C. N. Reynolds: "On the zeros of solutions of linear differential equations of the fourth order."

(7) Professor J. E. Rowe: "Related invariants of two rational sextics."

(8) Professor W. W. Johnson: "The nature and history of Napier's rules of circular parts."

(9) Professor O. E. Glenn: "On a new treatment of theorems of finiteness."

(10) Professor Louise D. Cummings: "The trains for the 36 groupless triad systems on 15 elements."

(11) Dr. Josephine R. Roe: "Interfunctional expressibility problems of symmetric functions (third paper)."

(12) Mr. B. F. GROAT: "Equations of the elastic catenary."

(13) Professor C. H. FORSYTH: "Relative distributions."

(14) Professor W. D. CAIRNS: "A derivation of the equation of the normal probability curve."

(15) Dr. MARY F. CURTIS: "Curves invariant under point transformations of special type."

(16) Professor G. D. BIRKHOFF: "On stability in dynamics."

(17) Professor DANIEL BUCHANAN: "Periodic orbits on a surface of revolution."

(18) Professor C. N. HASKINS: "Note on the roots of the function $P(x)$ associated with the gamma function" (preliminary communication).

(19) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group (third paper)."

(20) Mrs. CHRISTINE LADD-FRANKLIN: "Bertrand Russell and symbol logic."

Mrs. Ladd-Franklin was introduced by Professor Fiske. In the absence of the authors the papers of Professor Miller, Professor Daniell, Professor Mathewson, Mr. Reynolds, Professor Rowe, Professor Forsyth, and Dr. Schweitzer were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Given the system of equations

$$p_1 m_{11}^2 + 2p_2 m_{11} m_{21} + 2p_3 m_{11} m_{31} + p_4 m_{21}^2 + 2p_5 m_{21} m_{31} + p_6 m_{31}^2 = 0,$$

$$p_1 m_{11} m_{12} + p_2(m_{11} m_{22} + m_{12} m_{21}) + p_3(m_{11} m_{32} + m_{12} m_{31}) + p_4 m_{21} m_{22} + p_5(m_{21} m_{32} + m_{22} m_{31}) + p_6 m_{31} m_{32} = 0,$$

$$p_1 m_{12}^2 + 2p_2 m_{12} m_{22} + 2p_3 m_{12} m_{32} + p_4 m_{22}^2 + 2p_5 m_{22} m_{32} + p_6 m_{32}^2 = 0 \qquad (p_1 \neq 0);$$

to this system corresponds a determinant $|m_{ij}|$ $(i, j = 1, 2, 3)$. Dr. Robinson has demonstrated that, unless the relation

$$(p_2^2 - p_1 p_4)(p_3^2 - p_1 p_6) - (p_2 p_3 - p_1 p_5)^2 = 0$$

exists, the solutions of the given system of polynomials annul all the determinants of the matrix

$$\left\|\begin{matrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \end{matrix}\right\|$$

An analogous theorem holds for systems of polynomials which correspond to the determinant $|m_{ij}|$ $(i, j = 1, 2, \ldots, n)$.

2. When two operators s and t of a group satisfy the condition $s^{-1}ts = t^{-1}st$, they are of the same order α. If they are not otherwise restricted, Professor Miller shows that they generate a group of order $\alpha(2^\alpha - 1)$ which has a cyclic commutator subgroup of order $2^\alpha - 1$, and is generated by this subgroup and an operator of order α which transforms each of its operators into its square. There is one and only one such group for every positive integral value of α and this infinite system may be called the equitransform system of groups. When $\alpha = 2$ there results the symmetric group of order 6; when $\alpha = 3$, the semi-metacyclic group of order 21; etc. Each group of the system contains a set of $2^\alpha - 1$ conjugate cyclic subgroups of order α and all its operators of order 2 are conjugate whenever such operators occur.

3. In this paper Professor Daniell defines the derivative with respect to a function of limited variation, proves a modification of Vitali's theorem, and finally proves that, under certain restrictions, the original function is the integral of its derivative. A law for integration by parts is found and it is proved that any Stieltjes integral can be expressed as a single Stieltjes integral with respect to a non-decreasing function. The paper is to appear in the *Transactions*.

4. In this paper, Mr. Groat discusses the proper scale ratios to employ to secure similarity of flow of fluids in geometrically similar channels. The determinations are related to theory developed by him in a paper soon to appear in the *Transactions of the American Society of Civil Engineers*, volume 82.

It is shown that similarity of flow requires a scale ratio $L = N^{2/3} \div G^{1/3}$, where L = scale ratio, N = ratio of corresponding kinematic viscosities, and G = ratio of homologous accelerations,—all dimensionless ratios. The similarity then extends to details of the flow, including forces due to viscosity.

Attention is called to the fact that in ordinary model experiments G is unity. Therefore $L = N^{2/3}$. This shows that perfect similarity requires different values of the kinematic viscosity, which in turn requires either different fluids or different viscosity conditions of the same fluid in model and prototype.

Lamb (Hydrodynamics, page 537) states that the kinematic viscosity of air varies inversely as the pressure. If this is true, the sizes of model aeroplane propellers, for example, may be determined for various conditions as follows:

Pressure in atmospheres	1000	64	27
Value of N	1/1000	1/64	1/27
Scale ratio of model	1/100	1/16	1/9

This assumes, of course, that velocities do not approach too closely the velocity of sound in the air or other fluid involved.

The same effect might be secured by employing another fluid, as mercury, for example. The following table gives approximate ratios in two cases for different conditions: (*a*) model in water of prototype in air; (*b*) model in mercury of prototype in water.

Temperature (C.)		Temperature of water (C.) 10°	15°	25°
10°	Air	.204	.186	.158
	Mercury		.219	.255
20°	Air	.190	.174	.148
	Mercury		.220	.257
40°	Mercury		.210	.246

It is of importance to note that the scale ratio of a model in mercury of prototype in air can be obtained by taking the product of any two ratios for an intermediate field. Thus, the scale ratio for air at 20° and mercury at 40° can be secured by taking the product of the air-water and water-mercury ratios for water at any given temperature, say 15°, thus: $L = .174 \times .210 = .036$. Therefore a model aeroplane propeller of about 1/27 full size can be tested in mercury at 40° C., the air being at 20° C., with caution as to the velocity of sound in the media.

5. Moore has shown that the group of isomorphisms of an abelian group of order p^m, and type $(1, 1, \ldots, 1)$ is the linear congruence group,* Miller has considered the automorphisms

* *Cf.* also Burnside, Theory of Groups (1897), §§ 171, 172 and Chap. XIV.

of an abelian group of order p^m, type $(m - 1, 1)$,* and Ranum has developed the group of isomorphisms of any given abelian group of order p^m through a consideration of the group of classes of congruent matrices.† In Professor Mathewson's paper the viewpoint is different and the groups are treated wholly as abstract groups. The object is to study the groups of isomorphisms of the system of abelian groups of order p^m and type $(n, 1, 1, \ldots, 1)$, $n > 1$, and to show that these groups of isomorphisms may be built upon the group of isomorphisms of an abelian group of order p^m and type $(1, 1, \ldots, 1)$ or that the group of isomorphisms of the system under study ($n = 2, 3, 4, \ldots$) is an extension of the one of the system just before it and of index p in it.

6. Professor Birkhoff has proven theorems of oscillation and comparison for the solutions of ordinary linear differential equations of the third order, which are analogous to Sturm's well-known theorems concerning the solutions of linear differential equations of the second order (*Annals of Mathematics*, volume 12 (1911)). In the present paper Mr. Reynolds has proven similar theorems concerning the solutions of self-adjoint linear differential equations of the fourth order, and an oscillation theorem for the general linear differential equation of the fourth order.

7. Professor Rowe's paper appeared in the October BULLETIN.

8. In this paper Professor Johnson gives an historical survey of Napier's rules of the circular parts of right-angled spherical triangles. The principle underlying the rules is examined, and its partial anticipation as found in the early history of trigonometry. Some other cases of circular parts are also discussed.

9. An abstract of Professor Glenn's paper appeared in the March (1918) number of the BULLETIN, in connection with a preliminary report upon the subject, presented by title at the December meeting of the Society. Material since added includes a determination of the complete systems of orthog-

* Miller, *Transactions Amer. Math. Society*, vol. 2 (1901), pp. 259–264.
† Ranum, *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 71–91.

onal and of boolean invariants and covariants of the binary quintic.

10. In Professor Cummings' paper the sets of trains for the 36 groupless triad systems on fifteen elements are determined, and all types of trains which may occur in the 80 non-congruent systems Δ_{15} are exhibited.

11. In this paper, which is a continuation of those presented at meetings of the Society in October, 1916, and September, 1917, Dr. Roe formulates laws for zero coefficients in interfunctional expressibility tables of symmetric functions and proves triangularity as a property of certain of these tables.

12. Mr. Groat's second paper deals with equations of the elastic catenary deduced about the year 1902. The applicability to suspended cables was established at the time by taking transit observations upon ropes of a cable-way. Since then the theory has been expanded into a treatise adapted to practical engineering, which, though not yet published, has been employed upon transmission-line design and construction.

If Young's modulus be introduced, the following equations result:

$$\begin{aligned} u + c_0 &= \theta + m \sinh \theta, \\ v + c_1 &= \cosh \theta + \tfrac{1}{2} m \sinh^2 \theta, \\ w + c_2 &= \sinh \theta + \tfrac{1}{2} m (\sinh \theta \cosh \theta + \theta), \end{aligned}$$

where $u = x \div c$, $v = y \div c$, $w = z \div c$; u, v, and x, y, being similar systems of rectangular coordinates, and w, z, the corresponding rectifications of the similar curves from vertex to point u, v or x, y. The parameters are m and c, while θ is an independent variable which may be eliminated from any pair of the three equations, leaving a transcendental; for example, the rectangular equation of the elastic catenary.

Moreover, by employing in place of Young's modulus one in which the actual intensity of stress upon the current sectional area replaces that upon the original sectional area of the unstretched cable, another rectangular equation may be derived which is less cumbersome and fully as exact as the former, for all stresses below the elastic limit. There is evidence to show that such a modulus is more nearly constant in actual experiments than Young's.

With certain minor approximations the resulting equations are:

$$mv = \log[1 + m \cosh (\mu u)],$$

$$mw = u - 2 \tanh^{-1}\left[\sqrt{\frac{1-m}{1+m}} \tanh (\mu u)\right],$$

where $\mu = \sqrt{1 - m^2}$ and m and c have the same meaning as before, i. e., $m = H \div T$, $c = H \div s$, H being the total tension at the vertex, s the constant weight per unit length of unstretched cable, and T a constant depending upon the size and elastic properties involved.

Thus, it is possible to discuss any elastic catenary by either method as regards the effects of different loadings, temperatures, and other conditions, and this has been accomplished in numerous cases. Practical processes are simplified by employing different parts of the two methods in combination, the results being substantially the same by either, though obtained with greater facility by one or the other.

A general process for elastic curves of all kinds is pointed out, specifically that for an "elastic parabola."

13. In Professor Forsyth's paper a method is derived for making a series of interpolations in a systematic manner in a frequency distribution or a sequence of values separated by equal intervals to give a new frequency distribution or sequence of values separated by equal intervals which may or may not be the same as the original intervals. Thus, for illustration, given the population of a country such as the United States at the middle of the first of each year of several decennial periods, the method may be used to determine the population at the beginning of each such year.

14. The symmetrical distribution of magnitudes about their mean is commonly represented by a "polygon" whose equally spaced ordinates are proportional to the terms of the expansion of $(1 + 1)^n$. A direct proof, using Stirling's formula, is given by Professor Cairns of the theorem frequently quoted without proof in texts that, as n is increased indefinitely, the polygon (with a finite middle ordinate) approaches as its limiting form the normal curve $y = ke^{-h^2x^2}$. The method consists essentially in controlling what may be called the points of in-

flexion of the polygons so that these points approach predetermined positions on each side of the mean.

15. In his discussion of the curves defined by a differential equation of the form $dx/X = dy/Y$ Poincaré has investigated the curves invariant under the transformation

$$\tau: \quad \begin{aligned} x_1 &= S_1 x + \sum_{i+j=2}^{\infty} a_{ij} x^i y^j, \\ y_1 &= S_2 y + \sum_{i+j=2}^{\infty} b_{ij} x^i y^j, \end{aligned}$$

where $S_1 > 1 > S_2 > 0$, $S_2 > 1 > S_1 > 0$; and has shown that there always exist two invariant analytic curves through the invariant point (x_0, y_0). M. Lattes, using Poincare's methods, has shown that, when S_1, S_2 are distinct and not zero and one is not a positive integral power of the other, the same result holds. Lie's theory of infinitesimal transformations applied to the case $S_1 = S_2 = 1$—one of the cases in which Poincaré's methods fail—enables Dr. Curtis to determine the number of curves formally invariant under τ through the origin.

16. Professor Birkhoff's paper is devoted to questions concerning the stability of periodic orbits in dynamical systems with two degrees of freedom. The following theorem forms an important part of this investigation:

Given two closed curves C_1, C_2 about a point O in their plane, each cut by every radial half line from O once and only once, and given a one-to-one, direct, continuous, area-preserving transformation T of the ring C_1C_2 into itself, such that the image of any radial line under T is cut by any radial line at most once; then there exists an infinite connected set of invariant curves C about O, each cut by every radial line once and only once.

The results concerning stability will form part of a paper to appear in the *Proceedings of the American Academy of Arts and Sciences*.

17. In this paper Professor Buchanan determines the periodic orbits described by a particle which moves on a smooth surface of revolution, the axis of which is vertical. Periodic orbits are first determined for the vertical paraboloid of revo-

lution in much the same way as the well-known problem of the spherical pendulum is solved. Then the analytic continuation of these orbits is made with respect to a certain parameter which is introduced to differentiate the more general surface of revolution from the paraboloid.

18. Bourguet showed* that the function

$$P(z) \equiv \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} \frac{1}{z+\nu}$$

has two real roots in each of the intervals $-2n-2 < x < -2n-1$, $n \geqq 3$, and that these roots are of the form $-2n-1-\xi_n$, $-2n-2+\eta_n$, where ξ_n is of the order of $1/(2n)!$. Gronwall has recently shown† that $1/(2n)! < \xi_n < 6/(2n)!$, $1/(2n+1)! < \eta_n < 6/(2n+1)!$.

Professor Haskins points out that, though Bourguet's paper contains minor inaccuracies, typographical and other, his approximation is closer than that of Gronwall from $n = 8$ onward and gives the result that $\xi_n < (3+\epsilon_n)/(2n)!$, where $\lim_{n \doteq \infty} \epsilon_n = 0$.

Formulas given but not developed by Bourguet lead to an approximation equalling that of Gronwall for $n \geqq 2$ and with the limiting value $3/(2n)!$ for large n. Extension of Bourguet's methods leads to still closer approximation, e. g., $2.76/4! < \xi_2 < 2.91/4!$

19. Dr. Schweitzer constructs a table of undefined relations for the postulates for an abstract group, the relations being connoted by a functional notation. The first set of 2^2 undefined relations is of the second degree and has been treated previously.* The kth set of 2^{k+1} undefined relations is of the $(k+1)$st degree and is obtained by induction from the set for $k = k_0$; e. g., for $k = 3$ the indefinables are, $F_{13}(x_1x_2x_3)$, $F_{23}(x_1x_2^{-1}x_3)$, $F_{33}(x_1^{-1}x_2x_3)$, $F_{43}(x_1^{-1}x_2^{-1}x_3)$, $F_{53}(x_1x_2x_3^{-1})$, etc. Conceivably in the case of k variables one has $\Sigma_{s=1}^{m}\, {}^{m}C_s$ classes of systems of postulates for an abstract group, where the sub-

* Bourguet, L., *Comptes Rendus*, vol. 96 (1883), pp. 1307–1310; 1457–1490.

† Gronwall, T. H., *Annales Scientifiques de l'Ecole Normale Supérieure*, series (3), vol. 33 (1916), pp. 381–393.

* Compare abstracts, BULLETIN, May, 1918, p. 371, June, 1918, p. 428. In the former abstract, the relations II_2, II_3, II_1, should be stated as follows: II_2. $\phi\{f(x, y), y\} = x$, II_3. $f\{\phi(x, y), y\} = x$, II_1. $f\{f(y, x), f(z, x)\} = f(y, z)$.

script s denotes the number of indefinables involved in a system of postulates and $m = 2^{k+1}$.

In the present paper the author constructs for $k = 2$, five classes of systems of postulates for a group, finite or infinite, based on three and four indefinables respectively; also eight systems of postulates are constructed for $k = 3$, each system involving a single indefinable, $F_{13}, \ldots, F_{83}$ respectively. Infinitudes of new equations in iterative compositions are given (with special reference to the preceding 2^{k+1} indefinables) and their interpolation as functional equations is investigated. These iterative relations are found in partial solution of the problem: To find all equations in iterative compositions of order $w = 1, 2, 3, \ldots$, satisfied by the elements of an abstract group, finite or infinite.

20. Dr. Ladd-Franklin maintains that one reason why the great work of Bertrand Russell—the reduction of all mathematics to a few pure-logic "primitives" (the word here proposed to cover at once indefinable terms and undemonstrable propositions)—has not been more widely accepted than it has is because his views are, hitherto, somewhat subject to change. His first book, the Foundations of Geometry, has long since been substantially discarded. Few who read the Principles of Mathematics (1903) realize the extent to which it has already been superseded by the Principia Mathematica (1910). In this interval it has been found necessary to give up not only the whole doctrine of classes but also that of propositional functions and even of relations in general. Insurmountable difficulties which presented themselves have been solved, at present, by the drastic "no classes theory," though it was thought at first (see *Proceedings of the London Mathematical Society*, 1905, for the very illuminating discussion) that the zigzaggedness theory or the small classes theory might serve the purpose better. One naturally waits a bit longer to see if the "no classes theory" may not also prove to be fallible.

A defect that certainly still remains in the doctrine of Peano and Russell—an example of the several infelicities of their form of symbol logic—is the so-called new relation "epsilon"; there is nothing peculiar in the *relation* concerned—the specificity lies simply in the subject term, which is "individual" or singular. The only reason they give for regarding the relation as peculiar—that it is not subject (as is the relation of implication in general) to the rule of syllogism—is wholly fallacious; the real source of the difference lies in the change of sense of the middle term; that to confound in a middle term the sensus compositi and the sensus divisi is a source of danger has been a commonplace of logic since the time of the scholastics. That an inept symbolism is made use of in mathematics, which has for a fundamental interest the point and the "variable" (i. e., individuals, or singulars) would be of no consequence, but Russell and Peano treat this "addition" as constituting an important improvement over the logic which preceded them—that of Peirce and his school—instead of which it is simply erroneous.

F. N. COLE,
Secretary.

ON THE HEINE-BOREL PROPERTY IN THE THEORY OF ABSTRACT SETS.

BY DR. E. W. CHITTENDEN.

(Read before the American Mathematical Society, October 26, 1918.)

O. VEBLEN and N. J. Lennes have shown that in the presence of certain linear order axioms the Heine-Borel property is equivalent to the Dedekind cut axiom.* O. Veblen† and R. L. Moore‡ have used this property in systems of axioms for geometry and analysis situs.

M. Fréchet established the theorem that in a class (V) normale a subclass $\mathfrak{Q}$ has the Heine-Borel property if and only if $\mathfrak{Q}$ is compact and closed.§ This result was extended to systems ($\mathfrak{Q}$; K^{1367}) by T. H. Hildebrandt.||

E. R. Hedrick calls attention to the fact that the Heine-Borel theorem, in the enumerable case, is a consequence of the closure of derived classes.¶

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* Cf. O. Veblen, "The Heine-Borel theorem," this BULLETIN, vol. 10 (1904), p. 436–439.

† "A system of axioms for geometry, "*Transactions Amer. Math. Society*, vol. 5 (1904), pp. 343–384.

‡ "On the foundations of plane analysis situs," ibid., vol. 17 (1916), p. 131.

§ Sur quelques points du calcul fonctionnel," *Rendiconti di Palermo*, vol. 22 (1906), p. 26.

|| "A contribution to the foundations of Fréchet's calcul fonctionnel," *Amer. Journal of Math.*, vol. 34 (1912), pp. 281–282.

¶ "On properties of a domain for which the derived set of any set is closed," *Transactions Amer. Math. Society*, vol. 12 (1911), pp. 285–294.