index 8 whenever $n > 4$, and that the corresponding quotient group is the octic group. Hence K involves exactly three subgroups of index 2 whenever n exceeds 4 and only two other invariants subgroups besides identity, viz., the mentioned subgroup of index 8 and one of index 4 corresponding to the invariant subgroup of order 2 of the octic group. These results apply also to the special case when $n = 3$.

TRANSLATION SURFACES IN HYPERSPACE.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society, April 27, 1918.)

1. If the rectangular coordinates of the points of a surface can be expressed in the parametric form

$$(1) \qquad x_i = f_i(u) + g_i(v) \qquad (i = 1, 2, \ldots, n),$$

where f_i are functions of u alone and g_i functions of v alone, the surface is called a translation surface. It is seen that a translation can be found which will send any parameter curve $u =$ const. into any other one of the same system. The same is true of the curves $v =$ const. The surface (1) is also seen to be the locus of the mid-points of the lines joining the points of

$$(2) \quad C_1: \; x_i = 2g_i(u) \quad \text{to the points of} \quad C_2: \; x_i = 2f_i(v).$$

The character of the surface can then be determined, in a great measure, by the form and relative position of these two curves. Nearly all writers on surface theory* mention three facts concerning translation surfaces in 3-space:

(*a*) The generators of the developable which touches the surface along a curve $u =$ const. are tangent to the curves $v =$ const., or in other words the directions of the parameter curves passing through a given point are conjugate directions.

(*b*) There are surfaces which can be expressed in more than one way in the form (1).

* Darboux, Théorie générale des Surfaces, vol. 1, pp. 148, 340. Scheffers, Theorie der Flächen, vol. 2, pp. 188, 245.

(*c*) Minimum surfaces are always translation surfaces. The curves $u=$ const., $v=$ const., in this case, are minimum curves.

It is the object of the present note to examine translation surfaces in hyperspace as to the properties (*a*), (*b*), (*c*) and also to determine the relation between the curves C_1 and C_2 in order that (1) be a developable surface.

From (1) we see that the coordinates of the surface satisfy the linear partial differential equation of the second order

$$\frac{\partial^2 Z}{\partial u \partial v} = 0. \tag{3}$$

Segre* showed that if the coordinates of a surface satisfy a linear partial differential equation of the second order, then there are two directions through each point having the property (*a*) of conjugate directions. These are called the *characteristics* and are determined by the characteristic equation of the partial differential equation. The characteristic equation of (3) is

$$du dv = 0.$$

Therefore the parameter curves on (1) are characteristics. Hence *if* (3) *is the only partial differential equation of the second order which the coordinates of the surface satisfy, there are just two characteristics through each point and the surface cannot be expressed in more than one way in the form* (1). The form of equation (1) shows that the developables mentioned in (*a*) are cylinders.

2. *Ruled Translation Surfaces.*—Segre showed, in the paper referred to, that if the coordinates of a surface satisfy two linear partial differential equations of the second order the surface must either lie in a 3-space or else consist of the tangent lines to a twisted curve. I showed† that if the surface does not lie in a 3-space the two differential equations must be such that the pencil formed by them will contain only one equation of parabolic type. If now in addition to (3) the coordinates

* "Su una classa di superficie degl'iperspazii," ecc. *Atti di Torino*, 1907.

† C. L. E. Moore, "Surfaces in hyperspace which have a tangent line with three-point contact passing through each joint." This BULLETIN, vol. 18 (1912).

satisfy a second equation, it must be of the form

$$A\frac{\partial^2 Z}{\partial u^2}+B\frac{\partial Z}{\partial u}+C\frac{\partial Z}{\partial v}=0. \tag{4}$$

Every equation of the pencil

$$A\frac{\partial^2 Z}{\partial u^2}+\lambda\frac{\partial^2 Z}{\partial u\partial v}+B\frac{\partial Z}{\partial u}+C\frac{\partial Z}{\partial v}=0,$$

where λ is the parameter, will then be satisfied. The surface will then have an infinite number of characteristics. It is to be observed, however, that the characteristic equation is

$$Adv^2-\lambda dudv=0,$$

which for all values of λ has $dv=0$ for one factor. This is the direction of the characteristic of (4), and I showed in the article referred to that this is the direction of three-point contact and that there is only one such direction passing through a point and hence this must be the direction of the rulings of the developable. We therefore conclude that however the surface is expressed in the form (1) the rulings must form one parameter system. The other parameter system can be any one-parameter family of curves traced on the developable. Equation (1) then takes the form

$$x_i=k_iu+g_i(v),$$

where k_i are constants. It is seen that the rulings are parallel and therefore the surface is a cylinder. Hence *cylinders are the only translation surfaces in hyperspace which can be expressed in the form* (1) *in more than one way.*

The coordinates of any ruled surface must satisfy a parabolic differential equation of the second order, and if it is a translation surface will satisfy two equations and therefore will be the same as above. Hence *cylinders are the only ruled surfaces of translation.*

3. *Developables.*—Let u and v be the arc length on the parameter curves. Then the element of arc on the surface becomes

$$ds^2=du^2+a_{12}dudv+dv^2,$$

where $a_{12} = \Sigma f_i' g_i'$. The formula for the Gaussian curvature then reduces to

$$\sqrt{a}G = \frac{\partial}{\partial v}\left[\frac{2}{\sqrt{a}}\frac{\partial a_{12}}{\partial u}\right], \tag{5}$$

where $a = 1 - a_{12}^2$. If $G = 0$, we have on integrating

$$a_{12} = \cos(U + V), \tag{6}$$

where U is an arbitrary function of u alone and V an arbitrary function of v alone. This is then the condition that the surface be developable, that is, have zero curvature. From the definition of a_{12} we see that (6) says that the angle between any tangent to C_1 and any tangent to C_2 is always expressible as a function of u plus a function of v. To determine the relation between C_1 and C_2 in order that (6) may be satisfied let γ_1 and γ_2 be the spherical representation of the curves C_1 and C_2. (That is, γ_1 and γ_2 are the traces on the unit hypersphere, center at O say, made by lines through O parallel to the tangents.) The distance from a point of γ_1 to a point of γ_2, measured on the sphere, will be equal to the angle between the corresponding tangents to C_1 and C_2. Let u_i represent a series of points on γ_1 and v_i a series of points on γ_2. Let θ_{ij} represent the distance from u_i to v_i. Then, if relation (6) is satisfied, we have

$$\theta_{ri} = U(u_r) - V(v_i), \quad \theta_{si} = U(u_s) - V(v_i).$$

Subtracting,

$$\theta_{ri} - \theta_{si} = U(u_r) - U(u_s).$$

Thus the difference of the distances from v_i to u_r and u_s is independent of v_i and hence v_i must lie in the locus of points the difference of whose distances from u_r and u_s is constant. This must be true for any values of r and s. Hence if γ_1 is an n-dimensional curve (does not lie in a space of lower dimensions), γ_2 must reduce to a point, which means that C_2 is a straight line. If however γ_1 lies in a linear space $S_k (k < n)$ passing through O, then (6) will be satisfied if γ_2 lies in the space S_{n-k} completely perpendicular to S_k. In this case the distance from any point of γ_1 to any point of γ_2 is $\pi/2$. C_1 and C_2 will then be any two curves lying in completely perpendicular spaces. In particular if γ_1 is a great circle on the hypersphere, then γ_2 could either lie in the completely perpendicular space or it could coincide with γ_1, in which case (6) would be satisfied. The curves C_1 and C_2 for this last condition become any curves lying in parallel planes. The surface of translation would then reduce to a plane. If now γ_1 lies in a linear space S_k' which does not pass through O and γ_2 lies in the space S_{n-k}' passing through O and completely perpendicular to S_k', then each point of γ_2 will be the same distance from all points of γ_1 but these distances will vary with the point of γ_2. In this case then the distance is a function of v only and (6) will be satisfied. The curve C_2 will then be any curve lying in a space of $n - k$ dimensions and C_1 will be such that any tangent will make the same angles with any line in this space. If the space of C_2 be taken as a coordinate space the two curves must then have the form

$$\begin{aligned} C_1 &: \; x_i = g_i(u) \qquad (i = 1, 2, \ldots, k), \\ &\quad\; x_j = a_j u \qquad (j = k+1, k+2, \ldots, n); \\ C_2 &: \; x_i = 0 \qquad (i = 1, 2, \ldots, k), \\ &\quad\; x_j = f_j(v) \qquad (j = k+1, k+2, \ldots, n), \end{aligned} \tag{7}$$

where a_j are constants. We then have the results: (α) *if* C_1 *and* C_2 *lie in parallel planes, the resulting translation surface is a plane;* (β) *if they lie in completely perpendicular planes, or* (γ) *if they have the form* (7), *the translation surface is a non-ruled developable.* (δ) *If one of the curves reduces to a straight line, the translation surface is ruled.*

The surface generated by the midpoints of the lines joining the points of the two curves

$$x_1 = 2a \cos u, \quad x_2 = 2a \sin u, \quad x_3 = x_4 = 0.$$
$$x_1 = x_2 = 0, \quad x_3 = 2b \cos v, \quad x_4 = 2b \sin v$$

is the rotation surface

$$x_1 = a \cos u, \quad x_2 = a \sin u, \quad x_3 = b \cos v, \quad x_4 = b \sin v$$

which is left invariant by all the rotations leaving the x_1x_2 and x_3x_4 planes invariant.*

* C. L. E. Moore, "Rotations in hyperspace," *Proceedings American Academy*, vol. 53 (1918).

4. *Covariant Derivatives.*—In discussing those properties of a surface that have to do with the normals or curvature of curves traced on the surface it is very convenient to make use of covariant derivatives instead of the ordinary second partial derivatives. If the surface is expressed in vector form, the covariant derivatives are always normal to the surface while the ordinary second partial derivatives are not. A second vector fundamental form, analogous to the second fundamental form in three dimensions can be written in terms of these covariant derivatives.* The vector equation of (1) is

$$\rho = \sum_{1}^{n} (f_i(u) + g_i(v))k_i,$$

where k_i are unit vectors parallel to the coordinate axes. Then

$$(8) \qquad m = \frac{\partial \rho}{\partial u} = \Sigma f_i'(u)k_i, \quad n = \frac{\partial \rho}{\partial v} = \Sigma g_i'(v)k_i.$$

If u and v are arc lengths along the parameter curves,

$$(9) \quad m^2 = m\cdot m = \Sigma f_i'^2(u) = 1, \quad n^2 = n\cdot n = \Sigma g_i'^2(v) = 1.$$

The ordinary second partial derivatives are

$$(10) \qquad p = \frac{\partial^2 \rho}{\partial u^2} = \Sigma f_i''(u)k_i, \quad q = \frac{\partial^2 \rho}{\partial u \partial v} = 0, \quad r = \frac{\partial^2 \rho}{\partial v^2} = \Sigma g_i''(v)k_i.$$

From these we obtain

$$(11) \qquad \begin{aligned} m\cdot p &= \Sigma f_i'f_i'' = 0, \quad m\cdot q = 0, \quad m\cdot r = \Sigma f_i'g_i'', \\ n\cdot p &= \Sigma f_i''g_i', \quad n\cdot q = 0, \quad n\cdot r = \Sigma g_i'g_i'' = 0. \end{aligned}$$

In terms of these we can write the covariant derivatives†

$$(12) \quad y_{11} = (m \times n) \times (m \times n \times p) = \frac{1}{a}\begin{vmatrix} m & n & p \\ m^2 & m\cdot n & m\cdot p \\ n\cdot m & n^2 & n\cdot p \end{vmatrix},$$

* Wilson and Moore, "Differential geometry of two dimensional surfaces in hyperspace," *Proceedings American Academy*, vol. 52 (1916).

† Wilson and Moore, loc. cit., pp. 337–338.

where $a = 1 - (\Sigma f'g')^2$. The derivative y_{12} is obtained by replacing p by q in (12) and y_{22} is obtained by replacing p by r. It is seen that $y_{12} = 0$. This is a sufficient condition that the plane of the indicatrix (the locus of the end of the curvature vector of normal sections of the surface at a given point) pass through the surface point. Translation surfaces therefore are what Wilson and Moore called of the four-dimensional type.

5. If now the curves C_1 and C_2 lie in completely perpendicular spaces,

$$m \cdot n = \Sigma f_i' g_i' = 0.$$

This is the only surface of translation on which the parameter curves are orthogonal. The vector p will lie in the plane of C_1, and r in the plane of C_2, therefore

$$m \cdot r = \Sigma f_i' g_i'' = 0, \qquad n \cdot p = \Sigma f_i'' g_i' = 0.$$

Hence we have for this type of developable

$$(13) \qquad y_{11} = p, \qquad y_{12} = 0, \qquad y_{22} = r.$$

Thus p and r are normal to the surface and since they are the curvatures of the parameter curves we see that the curves $u =$ const., $v =$ const. are geodesics on the surface (curvature lies in the normal plane). Hence *on a non-ruled developable translation surface on which the generating curves are everywhere orthogonal those curves are geodesics.*

The area of the indicatrix is (Wilson and Moore, page 333)

$$2a^{3/2}\mu \times \delta = a_{12}y_{11} \times y_{22} + (a_{22}y_{11} - e_{11}y_{22}) \times y_{12};$$

and since $a_{12} = y_{12} = 0$, the area is zero. The indicatrix then reduces to a linear segment. Since y_{11} and y_{22} do not coincide in direction, this linear segment does not pass through the surface point. The curves $u =$ const., $v =$ const. are the Segre characteristics, and we know that when the indicatrix reduces to a linear segment the characteristics will be orthogonal and that the end of the curvature vector of these curves are the ends of the linear segment to which the indicatrix reduces. The indicatrix subtends a right angle at the surface point.

The vector mean curvature is half the sum of the vector curvatures in two perpendicular directions passing through the point considered. This sum is independent of the pair of perpendicular directions chosen. Then in this case we can choose the parameter curves and we have

$$h = \tfrac{1}{2}\Sigma(f_i'' + g_i'')k_i,$$

from which we see that the locus of the end of the mean curvature vector is also a translation surface and in fact is a developable of the type of the original surface. The locus of the indicatrix as the point in question describes the whole surface is the locus of lines cutting the two curves

$$C_1'':\quad x_i = 2f''(u), \qquad C_2'':\quad x_i = 2g_i''(v).$$

If the mean curvature is perpendicular to the indicatrix,

$$\Sigma f_i''^2 - \Sigma g_i''^2 = 0,$$

hence the curvature of C_1 and C_2 must be the same at all points and both constant, that is, they must be equal circles. If they have the same center the surface will be a rotation surface.

6. The second kind of non-ruled developable arises when

$$\begin{array}{lll} C_1: & x_i = 2f_i(u) & (i = 1, 2, \ldots, k), \\ & x_j = 2a_j u & (j = k+1, k+2, \ldots, n); \\ C_2: & x_i = 0 & (i = 1, 2, \ldots, k), \\ & x_j = 2g_j(v) & (j = k+1, k+2, \ldots, n), \end{array}$$

and the equation of the surface will be

$$\rho = \sum_1^k f_i(u)k_i + \sum_{k+1}^n (a_j u + g_j(v))k_j.$$

In this case then we have

$$m\cdot p = 0,\ n\cdot p = 0,\ m\cdot q = 0,\ n\cdot q = 0,\ n\cdot r = 0,$$
$$m\cdot r = \Sigma a_j g_j''.$$

If these values are substituted in (12), we have

$$y_{11} = p,\quad y_{12} = 0,\quad y_{22} = r - (m\cdot r)m + (m\cdot n)(m\cdot r)n.$$

Hence the curves $u =$ const. are geodesics on the surface, but in general the curves $v =$ const. are not. If however $m \cdot r = 0$ both sets of parameter curves are geodesics. Integrating this relation we have

$$\Sigma a_j g_j' = F = \text{const.},$$

and from what we saw previously concerning the relation § 3 this would require that the curve C_2 be a line and that the surface be ruled. Hence if C_1 and C_2 do not lie in completely perpendicular spaces and the translation surface is not ruled, both parameter systems cannot be geodesics on the surface.

The indicatrix for this type of surface does not reduce to a linear segment.

7. *Surfaces for Which C_1 and C_2 Coincide.*—If the curves C_1 and C_2 coincide, the surface (1) becomes the locus of the midpoints of the secants of a fixed curve. The surface is entirely similar to the same case in 3-space, except that in 3-space the fixed curve is an asymptotic line on the surface. Here the curve lies on the surface and is the locus of points at which the characteristics coincide. This curve has the property of an asymptotic line on a surface in 3-space.* The osculating plane of the curve is tangent to the surface and the tangent lines to this curve have three-point contact with the surface.† This is the only such line on the surface.

8. *Minimum surfaces.*—If we use the minimum curves on a surface as parameter lines, the coefficients in the first fundamental form are

$$a_{11} = m \cdot m = 0, \qquad a_{22} = n \cdot n = 0, \qquad a_{12} = m \cdot n.$$

The element of arc then becomes

$$ds^2 = a_{11} du^2 + 2a_{12} du dv + a_{22} dv^2 = 2a_{12} du dv.$$

The general formula for the mean curvature in terms of the covariant derivatives is

$$2h = \Sigma a^{(rs)} y_{rs},$$

where $a^{(rs)}$ is the complement of a_{rs} in the determinant $|a_{ij}|$

* Segre, loc. cit.

† C. L. Moore, "Surfaces in hyperspace, etc.," BULLETIN, vol. 18 (1912).

divided by a. In this case then

$$a^{(11)} = 0, \qquad a^{(22)} = 0, \qquad a^{(12)} = \frac{1}{a_{12}}$$

and the formula for h becomes

$$2h = \frac{1}{a_{12}} y_{12}.$$

The vanishing of the vector mean curvature is the necessary and sufficient condition for a minimum surface. Then, if the surface is minimum,

$$y_{12} = 0.$$

In 3-space if the minimum lines are taken as the parameter curves on a surface, the condition that the surface be minimum is the vanishing of the second coefficient in the second fundamental form. In hyperspace we have the same condition with respect to the vector second fundamental form. From (12) this condition becomes

$$y_{12} = (m \times n) \cdot (m \times n \times q) = 0.$$

The dot product is that used by Wilson and Lewis* and the vanishing here requires that q lie in the plane $m \times n$, that is

$$m \times n \times q = 0.$$

This is equivalent to saying that the coordinates must satisfy the differential equation

$$(14) \qquad A \frac{\partial^2 Z}{\partial u \partial v} + B \frac{\partial Z}{\partial u} + C \frac{\partial Z}{\partial v} = 0,$$

which again amounts to saying that there is a linear relation connecting m, n, q

$$(15) \qquad Aq + Bm + Cn = 0.$$

Differentiating the relations $m^2 = 0$, $n^2 = 0$, we see that

$$m \cdot q = 0, \qquad n \cdot q = 0.$$

* "Space time manifold of relativity," *Proceedings Amer. Acad.*, vol. 48 (1912).

Multiplying (15) by m, we have

$$Cm \cdot n = Ca_{12} = 0.$$

Since $a_{12} \neq 0$, $C = 0$. Likewise multiplying by n we see that $B = 0$. Hence equation (14) becomes

$$A \frac{\partial^2 Z}{\partial u \partial v} = 0.$$

Hence, *the minimum surface is a surface of translation. The necessary and sufficient condition that a surface in hyperspace be a minimum surface is that the minimum lines on it are characteristics.*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SOME ALGEBRAIC CURVES.

BY DR. JAMES H. WEAVER.

(Read before the American Mathematical Society, April 28, 1917.)

IN the following paper two algebraic curves are set up and some of their singularities are discussed. The author believes them to be new. At least a search through considerable of the literature on curves has failed to reveal them.

I.

Let there be any two distinct points A and B. Let the line joining A and B be drawn, and let the distance $AB = c$. Let there be drawn through A a line l_1 making an angle θ with AB, and let there be drawn through B a line l_2 making an angle $n\theta$ with AB (n an integer). We also consider that AB, l_1, and l_2 are in one plane. Let the intersection of l_1 and l_2 be C. It is required to find the locus of C.

Let A be the origin and let AB be the x-axis. Then the equations of the lines l_1 and l_2 will be

$$(1)\ y = x \tan \theta, \qquad (2)\ y = (x - c) \tan (n\theta)$$

respectively.