ON THE EVALUATION OF THE ELLIPTIC TRANSCENDENTS η_2 AND η_2'.

BY PROFESSOR HARRIS HANCOCK.

WITH Halphen* write

$$f(s) = 4s^3 - g_2 s - g_3 = 4(s - e_1)(s - e_2)(s - e_3) = S,$$

where e_2 is a real quantity and e_1, e_3 are conjugate imaginaries such that $e_1 - e_3$ is a positive pure imaginary. The discriminant $\Delta = g_2^3 - 27g_3^2$ is here negative. The elliptic differential equation in this notation is $ds^2/du^2 = 4s^3 - g_2 s - g_3$, whose integral is

$$u = \int_s^\infty \frac{ds}{\sqrt{f(s)}}.$$

If the lower limit of this integral be considered as a function of the integral, that is of u, we may write $s = \wp u$.

As is well known, the quantity ω_2 is defined by

$$(1) \qquad \omega_2 = \int_{e_2}^\infty \frac{ds}{\sqrt{f(s)}}; \quad \wp\omega_2 = e_2, \quad \wp'\omega_2 = 0.$$

If further we write

$$f_1(s) = 4s^3 - g_2 s + g_3 = 4(s + e_1)(s + e_2)(s + e_3),$$

the quantity ω_2' is defined through the relations

$$(2) \qquad \frac{\omega_2'}{i} = \int_{-e_2}^\infty \frac{ds}{\sqrt{f_1(s)}}; \quad \wp\omega_2' = e_2, \quad \wp'\omega_2' = 0.$$

Note that ω_2' is a pure imaginary.

It may be shown that the three half-periods

$$\omega_1 = \tfrac{1}{2}(\omega_2 - \omega_2'), \quad \omega_2, \quad \omega_3 = \tfrac{1}{2}(\omega_2 + \omega_2'),$$

correspond to the quantities e_1, e_2, e_3 above, the relations being the same as are those for ω, $\omega'' = \omega + \omega'$, ω' in the usual notation where all three of the roots e_1, e_2, e_3 are real.

Write

$$\frac{d}{du}\zeta u = -\wp u,$$

* Fonctions Elliptiques, vol. 1, chap. 6.

note that $\zeta(-u) = -\zeta u$ and that the quantities η_1 and η_3 are defined through the relations $\zeta\omega_1 = \eta_1$ and $\zeta\omega_3 = \eta_3$. Further write

$$\eta_1 = \tfrac{1}{2}(\eta_2 - \eta_2'), \quad \eta_3 = \tfrac{1}{2}(\eta_2 + \eta_2').$$

In practically all the text books on this subject it is shown that the two half-periods ω_1 and ω_2 play with respect to the function ζu the same rôle, when the discriminant is negative, as is played by the half-periods ω and ω' when the discriminant is positive; and the constants η_1 and η_3 which correspond to these half-periods are connected with η_2 and η_2' by the same relations which connect ω_1, ω_3 with ω_2, ω_2'. The values of η_2 and η_2' may be derived as follows:

In my monograph on Elliptic Integrals, page 51, formulas (39) and (40), it is shown that

$$\text{(3)} \quad \int_\alpha^x \frac{dx}{\sqrt{(x-\alpha)[(x-\rho)^2+\sigma^2]}} = \frac{1}{\sqrt{M}}\,\text{cn}^{-1}\left[\frac{M-(x-\alpha)}{M+(x-\alpha)}, k\right],$$

where

$$k^2 = \frac{1}{2} - \frac{1}{2}\frac{\alpha-\rho}{M}, \quad M^2 = (\rho-\alpha)^2 + \sigma^2;$$

and

$$\text{(4)} \quad \int_x^\alpha \frac{dx}{\sqrt{(\alpha-x)[(x-\rho)^2+\sigma^2]}} = \frac{1}{\sqrt{M}}\,\text{cn}^{-1}\left[\frac{M-(\alpha-x)}{M+(\alpha-x)}, k'\right],$$

where

$$k'^2 = \frac{1}{2} + \frac{1}{2}\frac{\alpha-\rho}{M}; \quad k^2 + k'^2 = 1, \quad 2kk' = \frac{\sigma}{M}.$$

These two formulas may be made the foundation of the whole so-called Weierstrassian theory, with negative discriminant.

To save the reader the trouble of deriving them and simply as a verification, write

$$u = \text{cn}^{-1}\left[\frac{M-(x-\alpha)}{M+(x-\alpha)}\right]$$

or

$$\operatorname{cn} u = \frac{M - (x - \alpha)}{M + (x - \alpha)} = \cos\varphi, \text{ where } \varphi \text{ is put } = \operatorname{am} u.$$

It follows at once that

$$x - \alpha = M\frac{1 - \cos\varphi}{1 + \cos\varphi}, \quad dx = 2M\frac{\sin\varphi\, dx}{(1 + \cos\varphi)^2},$$

$$(x - \rho)^2 + \sigma^2 = \frac{2M^2(1 + \cos^2\varphi) + 2(\alpha - \rho)M\sin^2\varphi}{(1 + \cos\varphi)^2}.$$

These values substituted in (3) cause that integral to become

$$\frac{1}{\sqrt{M}}\int_0^{\phi}\frac{dx}{\sqrt{1 - k^2\sin^2\varphi}}, \quad \text{where} \quad k^2 = \frac{1}{2} - \frac{1}{2}\frac{\alpha - \rho}{M}.$$

The indicated transformation reduces the integral (4) to the corresponding Legendre normal form.

Note that when $x = \alpha$, then $\varphi = \frac{1}{2}\pi$. From (3) and (4) it follows at once that

$$\omega_2 = \frac{K}{\sqrt{M}}, \quad \omega_2' = \frac{iK'}{\sqrt{M}}, \tag{5}$$

where

$$K = \int_0^{\pi/2}\frac{dx}{\sqrt{1 - k^2\sin^2\varphi}}, \quad K' = \int_0^{\pi/2}\frac{dx}{\sqrt{1 - k'^2\sin^2\varphi}},$$

k^2 and k'^2 having the values already indicated.

If one wished to evaluate an elliptic integral of the second kind in which there appeared a denominator, as in the integrals (3) and (4), similar reductions would transform it into the corresponding Legendre normal form and its value could be taken at once from the Legendre tables.

If however the Weierstrassian formulas are introduced, we may proceed as follows: From the definition above

$$\zeta u = -\int \wp u du;$$

or, since $\wp u = s$ and

$$du = \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}},$$

we may write*

$$\zeta u = \int^{s} \frac{sds}{\sqrt{4s^3 - g_2 s - g_3}},$$

with suitable choice of the constant in the lower limit.

If we put $e_1 = \rho + i\sigma$, $e_3 = \rho - i\sigma$, it is seen that

$$\int_{e_1}^{e_3} \frac{sds}{\sqrt{S}} = \int_{\rho+i\sigma}^{\rho-i\sigma} \frac{sds}{2\sqrt{(s-e_2)(s-e_1)(s-e_3)}}$$

$$= \int_{\rho+i\sigma}^{\rho-i\sigma} \frac{sds}{2\sqrt{(s-e_2)[(s-\rho)^2+\sigma^2]}}.$$

Observe that the left-hand integral is

$$\zeta\omega_3 - \zeta\omega_1 = \tfrac{1}{2}(\eta_2 + \eta_2') - \tfrac{1}{2}(\eta_2 - \eta_2') = \eta_2'.$$

In the right-hand integral put $s = \rho - it$ and it follows that

$$(6) \qquad -\eta_2' = \int_{t=\sigma}^{t=\sigma} \frac{(\rho - it)idt}{2\sqrt{(3\rho - it)(t+\sigma)(\sigma - t)}}.$$

To evaluate this integral write with Lucien Lévy† (see page 95)

$$\frac{1}{\sqrt{3\rho - it}} = \alpha + i\beta.$$

We have at once

$$\frac{3\rho + it}{9\rho^2 + t^2} = \alpha^2 - \beta^2 + i2\alpha\beta,$$

or

$$\alpha^2 - \beta^2 = \frac{3\rho}{9\rho^2 + t^2}, \quad 2\alpha\beta = \frac{t}{9\rho^2 + t^2},$$

$$\alpha^2 + \beta^2 = \frac{1}{9\rho^2 + t^2}.$$

As the sign of α is arbitrary, we shall take it positive; while the sign of β, from the second of the formulas just written, is the same as the sign of t. This may be indicated by the notation ϵ_t.

* See for example Schwarz, Formeln, etc., p. 87.

† Précis élémentaire de la Théorie des Fonctions Elliptiques, published by Gauthier-Villars, Paris, 1898.

It follows at once that

$$\alpha = \frac{1}{\sqrt{2}} \frac{\sqrt{3\rho + \sqrt{9\rho^2 + t^2}}}{\sqrt{9\rho^2 + t^2}},$$

$$\beta = \epsilon_t \frac{\sqrt{\sqrt{9\rho^2 + t^2} - 3\rho}}{\sqrt{9\rho^2 + t^2}}.$$

Write $(\rho - it)i(\alpha + i\beta)$ in the form $\alpha t - \beta\rho + i(\alpha\rho + \beta t)$ and put $\eta_2' = A + iB$. From formula (6) it is seen that

$$- A = \int_{t=-\sigma}^{t=+\sigma} \frac{dt}{2\sqrt{2}} \frac{1}{\sqrt{9\rho^2 + t^2}\sqrt{\sigma^2 - t^2}}$$
$$\times \left[t\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} - \rho\epsilon_t \sqrt{\sqrt{9\rho^2 + t^2} - 3\rho} \right],$$

$$- B = \int_{t=-\sigma}^{t=+\sigma} \frac{dt}{2\sqrt{2}} \frac{1}{\sqrt{9\rho^2 + t^2}\sqrt{\sigma^2 - t^2}}$$
$$\times \left[\rho\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} + \epsilon_t \sqrt{\sqrt{9\rho^2 + t^2} - 3\rho} \right].$$

If we write A in the form

$$\int_{-\sigma}^{0} + \int_{0}^{\sigma}$$

and in either of these integrals make the substitutions $t = -\tau$, it is seen that the one is the negative of the other, so that $A = 0$. This was evident à priori, since $\eta_2' = \zeta(\omega_2')$ is a pure imaginary, ζ being an odd function and ω_2' a pure imaginary.

It also follows that

$$B = \frac{\eta_2'}{i} = - \int_{t=0}^{t=\sigma} \frac{dt}{\sqrt{2}\sqrt{9\rho^2 + t^2}} \frac{1}{\sqrt{\sigma^2 - t^2}}$$
$$\times \left[\rho\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} + t\sqrt{\sqrt{9\rho^2 + t^2} - 3\rho} \right].$$

In this expression put $9\rho^2 + t^2 = x^2$, and it becomes, if $t > 0$,

$$(7) \qquad \frac{\eta_2'}{i} = - \int_{x=3\rho}^{x=M} \frac{(x - 2\rho)dx}{\sqrt{2(x - 3\rho)(M^2 - x^2)}},$$

where

$$M^2 = 9\rho^2 + \sigma^2.$$

To evaluate the integral (7), write $x = -t + \rho$, and we have

$$(8) \qquad \frac{-\eta_2'}{i} = \sqrt{2}\int_{t=-2\rho}^{t=\rho-M} \frac{(t+\rho)dt}{2\sqrt{(t-t_1)(t-t_2)(t-t_3)}},$$

where

$$t_1 = \rho + M, \quad t_2 = -2\rho, \quad t_3 = \rho - M; \quad t_1 + t_2 + t_3 = 0.$$

Write

$$u = \int_t^\infty \frac{dt}{\sqrt{4(t-t_1)(t-t_2)(t-t_3)}},$$

so that $t = \wp u$, an elliptic function with half-periods, say, $\omega, \omega', \omega'' = \omega + \omega'$.

The above transformations, although somewhat complicated, are instructive and are due to Lévy, loc. cit., pages 96 and 97. In the next substitution, however, he makes a rather curious mistake: For du he writes the value just given and associated with it he writes $t = \wp(u + \omega'')$. This of course vitiates his results, which are repeated under the heading "Principal Formulas," page 221. The formula (8) becomes*

$$\frac{-\eta_2'}{i} = \sqrt{2}\int_{\omega''}^{\omega'} (\wp u + \rho)du$$
$$= \sqrt{2}[-\zeta\omega' + \zeta\omega'' + \rho\omega' - \rho\omega''] = \sqrt{2}(\eta - \rho\omega).$$

The values of η and η' expressed in the Legendre normal forms are given in most text-books; see for example Schwarz, loc. cit., page 34. Schwarz fails, however, to give the corresponding values of η_2 and η_2'.

It is seen that

$$\frac{-\eta_2'}{i} = \sqrt{2}\left[\sqrt{t_1 - t_3}\left(E' - \frac{t_1 K'}{t_1 - t_3}\right) - \rho\frac{K'}{\sqrt{t_1 - t_3}}\right]$$
$$= \sqrt{M}\left[2E' - \frac{M + 2\rho}{M}K'\right].$$

* Here again we must be careful regarding the sign of $dt/du = +\sqrt{}$. See, for example, Appell et Lacour, Fonctions Elliptiques, p. 72.

The integral in formula (8) may be again calculated as follows: Write*

$$\sin^2\varphi = \frac{M - x}{M - 3\rho},$$

so that

$$M - x = (M - 3\rho)\sin^2\varphi,$$

$$x - 3\rho = (M - 3\rho)\cos^2\varphi, \quad M + x = 2M[1 - k'^2\sin^2\varphi],$$

where

$$k'^2 = \frac{1}{2} - \frac{3\rho}{2M}, \quad dx = (3\rho - M)2\sin\varphi\cos\varphi\, d\varphi.$$

The integral in question becomes

$$\frac{1}{\sqrt{M}}\int_0^{\pi/2}\frac{(3\rho - M)\sin^2\varphi + M - 2\rho}{\sqrt{1 - k'^2\sin^2\varphi}}d\varphi$$

$$= \frac{2M}{\sqrt{M}}\int_0^{\pi/2}\frac{1 - k'^2\sin^2\varphi}{\sqrt{1 - k'^2\sin^2\varphi}}$$

$$- \frac{2M}{\sqrt{M}}\int^{\pi/2}\frac{d\varphi}{\sqrt{1 - k'^2\sin^2\varphi}} + \frac{M - 2\rho}{\sqrt{M}}K'$$

$$= 2\sqrt{M}E' - \frac{M + 2\rho}{\sqrt{M}}K', \text{ as above.}$$

Note that $e_2 = -2\rho$. It is seen at once from formulas (1) and (2) above that η_2' is had from η_2 by changing the sign of e_2 and multiplying the value of η_2 by $-i$ and putting k' for k. We thus have

$$\eta_2 = \sqrt{M}\left[2E - \left(1 - \frac{2\rho}{M}\right)K\right], \quad 2\rho = -e_2, \quad M^2 = 9\rho^2 + \sigma^2,$$

$$\eta_2' = -i\sqrt{M}\left[2E' - \left(1 + \frac{2\rho}{M}\right)K'\right],$$

$$\eta_2\omega_2' - \eta_2'\omega_2 = i[2EK' + 2E'K - 2KK] = i2\pi.$$

The relation to the right is, of course, Legendre's celebrated formula.

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* See formula (17), page 45, of my monograph mentioned above.

One can judge from the present paper the little value of the Weierstrassian formulas when the applications of the general theory are involved or whenever any kind of numerical computation is desired. One sees also how easy it is to introduce errors in the calculation. The book of Lévy already mentioned is in most respects an excellent work, certainly from the standpoint of applied mathematics, much of the material being new; but when a substitution involving an elliptic function in the Weierstrassian form is introduced, the book is not free from errors. For example, not to mention many inaccuracies, besides the mistake already cited, it is seen that on page 82 of Lévy's book $e_1 + e_2 + e_3 \neq 0$. The same error is found on page 156, while in the calculation of ζu, the functions introduced on page 104 are incorrectly given. At the end of my larger work, volume 1, the Weierstrassian functions are put in juxtaposition with those of the older theory and it is seen that thereby nothing new is added.

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ON PLANE ALGEBRAIC CURVES WITH A GIVEN SYSTEM OF FOCI.

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1. Let

$$\phi(u, v, w) = 0 \tag{1}$$

be a curve of class n, and

$$u\xi + v\eta + w\zeta = 0 \tag{2}$$

a line with the coordinates u, v, w, and $x' = \xi/\zeta$, $y' = \eta/\zeta$ current cartesian coordinates. Then

$$-1 \cdot \xi - i \cdot \eta + (x + iy)\zeta = 0 \tag{3}$$

is a line which passes through the point (x, y) and the circular point I with the slope i. The coordinates of (3) are $u = -1$,