with $X_{n_2}Y_{n_2}$. If this process is continued there will be obtained an infinite sequence of arcs $X_{n_1}Y_{n_1}$ $X_{n_2}Y_{n_2}$, $X_{n_3}Y_{n_3}$, $\cdots$ no two of which have any point in common. For each i, the arc $X_{n_i}Y_{n_i}$ contains, as a subset, an arc $W_{n_i}Y_{n_i}$ which lies between the circles $\bar{K}$ and K_1, except for the points W_{n_i} and Y_{n_i} which lie on K_1 and $\bar{K}$ respectively. There exist 1) on $\bar{K}$ an infinite sequence of distinct points Y', Y_1', Y_2', Y_3', $\cdots$, 2) on K_1 an infinite sequence of distinct points W', W_1', W_2', W_3', $\cdots$, 3) an infinite sequence of distinct arcs $W_1'Y_1'$, $W_2'Y_2'$, $W_3'Y_3'$, $\cdots$ all belonging to the set $W_{n_1}Y_{n_1}$, $W_{n_2}Y_{n_2}$, $W_{n_3}Y_{n_3}$, $\cdots$, such that Y' is the sequential limit point of the sequence Y_1', Y_2', Y_3', $\cdots$ and W' is the sequential limit point of the sequence W_1', W_2', W_3', $\cdots$. No two of the arcs $W_1'Y_1'$, $W_2'Y_2'$, $W_3'Y_3'$, $\cdots$ have a point in common. It easily follows that there exists a closed connected point set N, containing Y' and W', such that every point of N is a limit point of the point set constituted by the sum of the arcs $W_1'Y_1'$, $W_2'Y'_2$, $W_3'Y_3'$, $\cdots$. The point set N is a continuous set of condensation of the set M.

Thus the supposition that M is not connected "im kleinen" leads to a contradiction. It follows that M is a continuous curve.

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DERIVATIVELESS CONTINUOUS FUNCTIONS.

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THERE is no more interesting illustration of the refinement of geometric intuition through the influence of the arithmetization of mathematics than that presented by the history of functions of this type. No less a mathematician than Ampère, not to mention Duhamel and Bertrand, thought he had actually proved that continuous functions had derivatives for all save a finite number of arguments. Darboux in his paper on "Discontinuous functions" published in the *Annals* of the Ecole Normale for 1875, though dated January 20, 1874, in connection with his example of a function of this class mentions only one auditor, M. Bienaimé, who said that he was unconvinced by Ampère's proof. Darboux makes no mention of Weierstrass's work published in 1874 in *Crelle* on the function $\Sigma a^n \cos b^n x$ and was doubtless not aware of it. Du Bois-Reymond was so awestruck by Weierstrass's curve as to provoke some rather jocular remarks by Wiener in the introduction to his paper on Weierstrass's curve in the 90th volume of *Crelle's Journal.*

It does not seem to have been noticed by writers who have considered these functions, especially of Weierstrass's type, that a considerable simplification of treatment was possible by a more obvious choice of the δx used in the incremental ratio and that other advantages might result from such an innovation.

We begin by treating in detail a Weierstrass function

$$W(x) = \sum_0^\infty a^n \sin b^n \pi x \qquad |a| < 1,\ b \text{ integral.}$$

Setting

$$\delta x = 2k/b^{N+1}, \quad k \text{ integral},$$

we get by applying the mean value theorem to the first N terms and a trigonometric identity to the last term,

$$(1) \quad \frac{\Delta W(x)}{\delta x} = \pi \sum_0^{N-1} (ab)^n \cos b^n \pi (x + \theta \delta x) + \pi (ab)^N \sin \frac{k\pi}{\pi k/b} \cos \left(b^n x + \frac{k}{b} \right) \pi.$$

Evidently the absolute value of the first N terms is less than

$$\pi \sum_0^{n-1} |ab|^n < \frac{\pi |ab|^n}{|ab| - 1}$$

if $|ab| > 1$. As to the last term in (1), if we suppose that $k \,\|\!\wedge\, \tfrac{3}{4}b$, which implies that 4 is a factor of b, it is, in absolute value,

$$(2) \qquad \geqq \pi |ab|^N \frac{1}{\frac{3}{4}\pi \sqrt{2}} \left| \cos \left(b^n x + \frac{k}{b} \right) \pi \right|$$

In this expression we can find two values of k, k_1 and k_2, such that

$$1^\circ. \quad 1 \geqq \cos\left(b^n x + \frac{k_1}{b}\right)\pi \geqq \frac{1}{\sqrt{2}}$$

$$2^\circ. \quad -\frac{1}{\sqrt{2}} \geqq \cos\left(b^n x + \frac{k_2}{b}\right)\pi \geqq -1$$

where k_1 and k_2 have *opposite** signs.

Thus for these values of k

$$\left|\pi(ab)^N \frac{1}{\frac{3}{4}\pi\sqrt{2}} \cos\left(b^N x + \frac{k}{b}\right)\pi\right| \geqq \frac{\pi|ab|^N}{\frac{3}{2}\pi}.$$

The last term of (1) will thus dominate in sign and magnitude the first N terms if

$$|ab| > 1 + \tfrac{3}{2}\pi.$$

Hence the right and left incremental ratio which we are considering will become infinite with N but will always have opposite signs.

This proves that $W(x)$ has a derivative for no value of x.

Making use of the remark at the end of the footnote, we can prove that, except possibly for a set of x-points of Borel measure zero (null set), $W(x)$ has neither a backward nor a forward tangent.

To do this suppose x written in system radix b, i. e.,

$$x = b_0 + \frac{b_1}{b} + \frac{b_2}{b^2} + \cdots,$$

where b_1, b_2, etc., $< b$. All those x's in which any one of the digits $0, 1, \cdots, b-1$ fails to occur infinitely often form a null set. Hence, except for the points of the null set $[x]$

* This can be seen at once from a diagram. Divide the angle 2π into 8 equal parts. If for example $b^N x\pi$ lies in the first octant, one and possibly more values of k can be found between 1 and $\frac{3}{4}b$ so that 1° holds; while one or possibly more values lie between 0 and $-b/4$ for which 2° holds. Similarly for the other octants.

Moreover it is clear that if $b^N x\pi$ has its terminal line between $I\pi$ and $(I + 1/b)\pi$, I integral or zero, a positive k_1 can be found for which 1° holds and another positive k_2 for which 2° holds. If $b^N\pi x$ lies between $(I + (b-1)/b)\pi$ and $(I+1)\pi$, two negative k's can be found for which 1° and 2° hold respectively.

for which $b_i = 0$ and $b_i = b - 1$ fail to occur infinitely often, the right and left incremental ratios have each for their upper and lower limits $+\infty$ and $-\infty$ respectively. The set $[x]$ is made up of a countable number of perfect null sets and is everywhere dense on the line; it is consequently not closed.

At the points of $[x]$ $W(x)$ might possibly have vertical cusps but not elsewhere.

It will be noted that the signs of the individual terms of our series are of no significance and hence can be arbitrarily changed to $+$ or $-$. This conclusion does not seem to follow from Weierstrass's proof.

Weierstrass (also Dini and others) unnecessarily require that b be an odd integer; our restriction that b be a multiple of 4 can evidently be removed if $|ab| > 9$, and the proof just given will remain valid.

The formation of an extensive class of such functions

$$W(x) = \sum_0^\infty u_n(x) \sin i_n x\pi \quad \text{or} \quad \sum_0^\infty u_n(x) \cos i_n x\pi,$$

i_n integral, is easy. We require:

1°. Uniform convergence when $l \leqq x \leqq L$.

2°. i_n must divide i_{n+1} and for an unlimited number of n's their ratio must either be divisible by 4 or increase indefinitely with n.

3°. $\sum_0^\infty u_n'(x)$ must be uniformly convergent by Weierstrass's G test, $l \leqq x \leqq L$.

4°. $\frac{3}{2}\pi \sum_0^{N-1} |i_n u_n(x)| < |i_N u_N(x)|$ in the interval of convergence of $W(x)$.

These conditions are merely sufficient and 2° and 4° admit of certain obvious modifications. Functions which fulfill these conditions will have no derivatives in the interval (lL).

The following specimens will suffice:

1. $\sum_0^\infty \frac{a^n}{n!} \frac{\sin}{\cos} (n!\pi x) \qquad |a| > 1 + \frac{3}{2}\pi.$

2. $\sum_{1}^{\infty} \frac{a^n}{1\cdot 3\cdot 5\cdots(2n-1)} \cos\,(1\cdot 3\cdot 5\cdots 2n-1)\pi x$

$|a| > 1 + \frac{3}{2}\pi$ (Dini).

3. $\sum_{1}^{\infty} \frac{a^n}{1\cdot 5\cdot 9\cdots(4n+1)} \sin\,(1\cdot 5\cdot 9\cdots(4n+1))\pi x$

$a > 1 + \frac{3}{2}\pi$ (Dini).

4. If $\sum_{1}^{\infty} \frac{a_i}{10^i}$ denote any non-terminating decimal,

$$\sum_{0}^{\infty} \frac{a_i}{10^i} \begin{matrix}\sin\\ \cos\end{matrix} (10^{3i} x\pi).$$

5. $\sum_{0}^{\infty} \frac{1}{a^n} \begin{matrix}\sin\\ \cos\end{matrix} (n!a^n \pi x)$, where $|a|$ is an integer > 1.

6. $x \sum_{0}^{\infty} a^n \sin b^n x\pi$, $|a| < 1$, $|ab| > 1 + \frac{3}{2}\pi$, has a derivative for $x = 0$ but for no other value of x.

7. $\sum_{0}^{\infty} \frac{x^n}{n!} \begin{matrix}\sin\\ \cos\end{matrix} (n!\pi x)$ has derivatives *between* -1 and $+1$ and no derivatives if $|x| > 1 + \frac{3}{2}\pi$.

Lerch gives a theorem* which shows that this last function has no derivatives for any *rational* points for which $|x| \geqq 1$. It is easy to show that it can have a finite derivative for no point $|x| > 1 + \frac{\pi}{2}$.

A HALF CENTURY OF FRENCH MATHEMATICS.

Les Sciences Mathématiques en France depuis un Demi-Siècle. Par Emile Picard. Paris, Gauthier-Villars, 1917. 24 pp.

In the first decades of the last century the home of the scientific spirit was in France. Paris was the capital of the Republic of exact truth. Interest in scientific discovery and creation was widespread among her people. The spirit of literature flourished alongside the spirit of exact researches

* Lerch, *Crelle's Journal*, vol. 103, p. 130 ("Ueber die Nichtdifferentiierbarkeit gewisser Funktionen").