THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and third regular meeting of the Society was held in New York City on Saturday, April 26, 1919, extending through the usual morning and afternoon sessions. This being the first eastern meeting since October, the attendance was large, indicating, as it may be hoped, a revival of the conditions preceding the war. The following sixty-seven members were present:

Professor R. C. Archibald, Professor R. A. Arms, Dr. Charlotte C. Barnum, Professor Susan R. Benedict, Professor G. D. Birkhoff, Professor B. H. Camp, Professor F. N. Cole, Professor Louise D. Cummings, Professor C. H. Currier, Dr. Mary F. Curtis, Dr. Tobias Dantzig, Dr. J. V. DePorte, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor A. B. Frizell, Mr. T. C. Fry, Professor W. V. N. Garretson, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor C. C. Grove, Professor C. O. Gunther, Captain Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. J. R. Kline, Dr. K. W. Lamson, President E. O. Lovett, Professor James Maclay, Professor L. C. Mathewson, Professor R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor G. W. Mullins, Dr. F. D. Murnaghan, Mr. L. S. Odell, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Susan M. Rambo, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor E. D. Roe, Jr., Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Dr. J. M. Stetson, Professor H. D. Thompson, Mr. H. S. Vandiver, Major Oswald Veblen, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Dr. C. E. Wilder, Miss Ella C. Williams, Dr. Emily C. Williams, Professor Ruth G. Wood, Professor J. W. Young.

President Morley occupied the chair, being relieved by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Mr. N. W. Akimoff, Philadelphia, Pa.; Dr. Tobias Dantzig, Columbia

University; Mr. A. C. Maddox, Guthrie, Okla.; Mr. Montford Morrison, Chicago, Ill.; Professor Ganesh Prasad, Central Hindu College, Benares, India; Mr. F. M. Weida, State University of Iowa; Mr. C. L. E. Wolfe, University of California. Two applications for membership in the Society were received.

It was decided to hold the summer meeting of the Society at the University of Michigan, on September 2–4. Professors Beman, Bliss, Karpinski, Osgood, and the Secretary were appointed a committee on arrangements for this meeting A committee, consisting of Professors Bliss, Birkhoff, and Veblen, was appointed to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

Professors E. W. Brown, L. E. Dickson, and H. S. White were appointed as representatives of the Society in the Division of Physical Sciences of the National Research Council, and President R. S. Woodward and Professors G. D. Birkhoff and W. D. MacMillan as representatives of the Society in the American Section of the International Astronomical Union.

Professor Archibald, chairman of the committee on the publication of a mathematical year book, presented a preliminary report; the committee was continued and asked to make a further report at a future meeting.

Professor A. B. Coble was appointed to succeed Professor D. R. Curtiss as a member of the Editorial Committee of the *Transactions*.

A special feature of the meeting was the reports on the work in ballistics at Aberdeen and Washington, which occupied the first part of the afternoon session. Titles and abstracts of these reports are included in the list below (papers 14, 15, 16).

About fifty members and friends gathered at the midday luncheon. Thirty-two attended the dinner in the evening. Much satisfaction was expressed at the return of these pleasant occasions.

The following papers were read at this meeting:

(1) Professor C. J. Keyser: "Concerning groups of dyadic relations in an arbitrary field."

(2) Mr. J. K. Whittemore: "Certain functional equations connected with minimal surfaces."

(3) Professor W. B. Fite: "Linear functional differential equations."

(4) Mr. L. B. Robinson: "Note on a theorem due to Wilczynski."

(5) Mr. L. B. Robinson: "A curious system of polynomials, continued."

(6) Professor O. E. Glenn: "Covariants of binary modular groups."

(7) Professor O. E. Glenn: "Modular covariant theory of the binary quartic. Tables" (preliminary report).

(8) Professor O. E. Glenn: "Invariants of velocity and of acceleration."

(9) Professor F. H. Safford: "Reduction of the elliptic element to the Weierstrass form."

(10) Mr. Philip Franklin: "Computation of the complex roots of the function $P(z)$."

(11) Dr. A. R. Schweitzer: "On the history of functional equations" (preliminary report).

(12) Professor E. D. Roe, Jr.: "The irreducible factors of $1 + x + \cdots + x^{n-1}$. Second paper."

(13) Professor E. D. Roe, Jr.: "The irreducible factors of a circulant."

(14) Captain Dunham Jackson: "Contributions of modern mathematics to exterior ballistics."

(15) Dr. T. H. Gronwall: "Qualitative properties of the ballistic trajectory."

(16) Major Oswald Veblen: "Progress in design of artillery projectile ."

(17) Professor G. D. Birkhoff: "Boundary value and expansion problem for differential systems of the first order."

(18) Professor G. D. Birkhoff: "Note on the closed curves described by a particle moving on a surface in a gravitational field."

(19) Professor G. D. Birkhoff: "Note on the problem of three bodies."

(20) Professor Edward Kasner: "A characteristic property of central forces."

(21) Dr. J. F. Ritt: "On weighting-factor curves for low elevations."

(22) Professor A. C. Lunn: "Some functional equations in the theory of relativity."

(23) Dr. J. R. Kline: "Concerning sense on closed curves in non-metrical plane analysis situs."

(24) Professor R. L. Moore: "On the most general class

L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true."

(25) Mr. H. S. VANDIVER: "On the class number of the field $\Omega(e^{2i\pi/p^n})$ and the second case of Fermat's last theorem."

(26) Professor F. W. BEAL: "On certain points of congruences of circles."

(27) Professor L. L. SILVERMAN: "Regular transformations of divergent series and integrals."

(28) Mr. T. C. FRY: "The application of the modern theories of integration to the solution of differential equations."

(29) Dr. C. A. FISCHER: "Completely continuous transformations and Stieltjes integral equations."

(30) Professor ARNOLD EMCH: "On closed curves described by a spherical pendulum."

(31) Professor H. S. WHITE: "An explicit formula for two old problems."

(32) Professor L. P. EISENHART: "Triply conjugate systems with equal point invariants."

Mr. Franklin was introduced by Dr. Gronwall. The papers of Mr. Robinson, Professor Safford, Dr. Schweitzer, Professor Lunn, Professor Beal, Professor Silverman, Mr. Fry, Dr. Fischer, Professor Emch, Professor White, and Professor Eisenhart were read by title. Abstracts of the papers follow below.

1. If the domain of a relation be identical with its codomain and therefore with its field, then, whatever field be given, any relation of the kind in question belongs to one and but one of the following nine types of relations: one-one, one-some, some-one, some-some, one-many, many-one, some-many, many-some, many-many. These classes together with those formed from them by logical addition constitute a certain class C of classes of relations. Professor Keyser has determined what members of C are groups, the rule of combination being relative multiplication.

2. In Mr. Whittemore's paper functional equations for the Enneper-Weierstrass function defining a real minimal surface, which express certain properties of the surface, are solved, and from the results several theorems are obtained.

3. Professor Fite establishes the existence of solutions of a

certain class of linear functional differential equations and discusses the properties of the functions defined by certain mixed difference equations.

4. At a previous meeting of the Society (April 28, 1917) Mr. Robinson showed that the general system of linear homogeneous differential equations could be reduced to a canonical form whose covariants could be computed by a finite number of quadratures. This result was accepted for publication by the editors of the *Johns Hopkins Circular*. Subsequently the author discovered that his method yields a very neat proof of the theorem enunciated by Wilczynski on page 39 of his Projective Differential Geometry.

5. In this paper Mr. Robinson continues a note read at the summer meeting of 1918. Given a system of equations

$$\sum_{j=1}^{n+1} P_{jj}x_{jr}^2 + 2\sum_{i=1}^{n+1}\sum_{k=1}^{n+1} P_{ki}x_{kr}x_{ir} = 0$$

$$(r = 1, 2, \cdots, n), \qquad (i > k);$$

polarize each of the above equations with respect to the r following points: $x_{1r}, x_{2r}, \cdots, x_{n+1\,r}$ $(r = 1, 2, \cdots, n)$. Denote the resulting system by J_2. The system composed of J_1 and J_2 can be designated by P. Solve J_1 with respect to x_{1r}, substitute the solutions in J_2, and rationalize. Write the coefficients of the resulting forms thus:

$$C_\lambda(P_{11} \cdots P_{n+1\,n+1}).$$

The necessary and sufficient condition for the existence of values of the x's annulling P and not annulling all the determinants of the matrix $\| x_{ij} \|$ $(i = 1, 2, \cdots, n + 1)(j = 1, 2, \cdots, n)$ is given by the vanishing of all the C's.

6. Contributions relating to processes, general and enumerative, for complete systems of covariants of binary forms transformed modulo 2, or 3, are treated in Professor Glenn's paper, which is a summary of the tabulated results, and methods, of a number of papers by the author on formal modular covariant theory (mod p).* An algorism of modular

* For a treatment of universal covariants of modular groups, see Dickson, *Transactions*, vol. 12 (1911), p. 75.

convolution, involving simultaneous invariants and covariants, is here given as a new conclusion, and the forms from the complete systems of former papers are exhibited under the notation of this algorism.

The paper appeared in the April number of the *Proceedings of the National Academy of Sciences.*

7. The doctrine of the covariant scale and of the construction of a complete formal modular concomitant system by a process of passing from degree to degree, developed by Professor Glenn in former papers, now enables him to derive fundamental systems of the binary quartic form with reference to the linear congruence groups G_6 (mod 2), and G_{48} (mod 3). He also extends the general theory of the forms of even order transformed modulis 2 and 3, and summarizes certain conclusions in this theory in covariant tables for the forms of orders 4, 6, 8. The problem of the complete seminvariant system is, in the methods of this paper, a phase of a theorem on the reducibility modulo p of the general quartic of order $> p^2 - 1$.

8. In relativity theory the velocity $(\dot{x}, \dot{y}, \dot{z})$ of a point moving uniformly, as observed from a system of reference S, is connected with the apparent velocity $(\dot{x}', \dot{y}', \dot{z}')$ for an observer situated upon a system S', moving relatively to S, by a set of equations of substitution upon the variables $\dot{x}, \dot{y}, \dot{z}; \dot{x}', \dot{y}', \dot{z}'$. Professor Glenn constructs systems of concomitants, both universal and quantitative, in each of two domains, for these transformations. A similar invariant theory of acceleration is developed.

9. A formula for the reduction of the elliptic element to the Weierstrass form and quoted by Biermann as derived from Weierstrass' lectures has been discussed briefly by Greenhill, Enneper and others. Haentzschel has referred to it as the most general solution of the differential equation satisfied by all elliptic functions, and then used it as a basis of extension and criticism of Wangerin's surfaces in a potential problem. Professor Safford has published several articles concerning Wangerin's and Haentzschel's results, and in this paper gives new facts about the formula, showing in particular that it is not a more general form but arises solely from a change of constants.

10. By a method based on an existence proof recently given by T. H. Gronwall, the numerical values of the four complex roots of the P-function associated with the gamma function have been calculated by Mr. Franklin. The values found are: $-1.7262976 \pm 1.2380921i$ and $-3.7264730 \pm .5406746i$.

11. Dr. Schweitzer compiles a first section of a bibliography (with critical notes) of functional equations, excluding references primarily to difference, differential and integral equations, or mixed types of such equations. The present work, which the author hopes to amplify, is divided into I. Classification; II. Range of application; III. Methods of solution. Subdivisions under I are I_1. Simple, and I_2. Composite genesis; I_3. Analytic, and I_4. Non-analytic reference, and under I_4 the further divisions I_{41}. General theories of composition, with the sections I_{411}. Equations in partial functions, and I_{412}. Equations in iterative compositions. The operations considered are mainly distribution, inversion, elimination, iteration, transformation as applied to variables or functions. Attention is called to the relations given by Hankel* and Stolz-Gmeiner† when interpreted as functional equations (cf. Hankel: "Calculus of Operations") and their generalization, and the interesting but apparently little known functional equations discussed by E. Schroeder.‡ In the *Mathematische Annalen*, volume 12, page 483, Schroeder gives an attempted, but erroneous, solution of a distributive equation in the domain of substitution groups.

12. In this paper, which is a continuation of a previous one, Professor Roe derives relations among the roots of $1 - x^n = 0$, various formulas for factorization, and the irreducible factors of $f(x, n) \equiv 1 + x + \cdots + x^{n-1} \equiv 1, n$, of $f(x^m, n) \equiv m, n$ and of $X_n(x^m) \equiv {}_n^m$. $X_n(x) = 0$ gives the primitive nth roots of unity, and $X_n(x) \equiv {}_n^1$ is irreducible. The paper establishes that an infinite number of identities each of two types comprises all the interfunctional expressibility relations among the f's and X's, namely:

$$S_i \equiv F(T_1, T_2, \cdots, T_e),$$

* Theorie des complexen Zahlensysteme (1867), pp. 26–28, 34, 13, 63, 106 and elsewhere.

† Theoretische Arithmetik (1902), chapter III, especially, pp. 52, 53, 38, note 1.

‡ For example, *Archiv* (2) vol. 5; *Crelle*, vol. 90; *Math. Annalen*, vols. 10, 29; and the Vorles. u. d. Algebra der Logik, vol. 1 (1890), pp. 617, 630 et seq.

which states that every S can be expressed as an explicit rational function of the T's, and

$$F_1(S_1, S_2, \cdots, S_r) \equiv F_2(T_1, T_2, \cdots, T_s).$$

In both identities S's and T's represent both of them X's, or both f's, or either one f's and the other X's, making eight subtypes in all. Their development and discussion form a large part of the paper, and they are used in solving the main problem.

The most general solutions are:

For irreducible factors of the X's

$$\frac{n_1^{a_1} \cdots n_r^{a_r} m_1^{c_1} \cdots m_j^{c_j}}{m_1^{b_1} \cdots m_k^{b_k}} \equiv \prod_{\Sigma\lambda_i=0}^{\Sigma\lambda_i=\Sigma a_i} n_1^{\lambda_1} \cdots n_r^{\lambda_r'} m_1^{\nu_1} \cdots m_k^{\nu_k},$$

$\nu_i = b_i + c_i$, $k \geqq j \geqq 0$, $a_i \geqq \lambda_i \geqq 0$, and the product contains $\Pi_1^r\ (a_i + 1)$ irreducible factors.

For irreducible factors of the f's

$$n_1^{a_1} \cdots n_r^{a_r} m_1^{c_1} \cdots m_j^{c_j},\ m_1^{b_1} \cdots m_k^{b_k}$$
$$\equiv \prod_{\Sigma\lambda_i=0,\ \Sigma\nu_i=1+\Sigma c_i}^{\Sigma\lambda_i=\Sigma a_i,\ \Sigma\nu_i=\Sigma(b_i+c_i)} n_1^{\lambda_1} \cdots n_r^{\lambda_r'} m_1^{\nu_1} \cdots m_k^{\nu_k},$$

$k \geqq j \geqq 0$, $c_{j+l} = 0$ when $l > 0$, $a_i \geqq \lambda_i \geqq 0$, $b_i + c_i \geqq \nu_i \geqq c_i$, a product that contains

$$\prod_1^r (a_i + 1) \left\{ \prod_1^j (b_i + c_i + 1) \prod_{j+1}^k (b_i + 1) - \prod_1^j (c_i + 1) \right\}$$

irreducible factors.

Numerous applications of the theory and methods are given. It is pointed out that a theorem of Lombardi is contained as a very special case in the second of the preceding general solutions, which is the generalization of all problems of the type of Lombardi's. A theorem proved by De Virieu is also generalized: The necessary and sufficient conditions that $f(x) \equiv \Pi_1^k(x^{\alpha_i} - 1)/\Pi_1^r(x^{\beta_i} - 1)$ is integral are: no prime number enters more β's than α's, and its distribution in the individual β's is not higher than in the individual α's. If $r = k = p$, $\alpha_i = m - i + 1$, $\beta_i = i$, and x is replaced by x^b, we get the theorem proved by De Virieu. The paper concludes with lists of irreducible factors and applications of the developed theory.

13. The observations contained in this paper will be incorporated in Professor Roe's paper preceding.

14. Captain Jackson's paper gives a brief summary of recent progress in the treatment of the fundamental problems of exterior ballistics under the guidance of modern methods in mathematical analysis. Particular emphasis is placed on Moulton's development of the method of numerical integration for computing trajectories and differential variations, and on Bliss's method of computing differential variations by the use of an adjoint system of differential equations. It is pointed out further that, while the theory of the motion of a projectile, regarded as a material particle, has reached a relatively satisfactory state, present knowledge of the accompanying gyroscopic and fluid motions is highly incomplete.

15. In this paper Dr. Gronwall considers the differential equations for the motion of a heavy particle in a resisting medium $x'' = -Fx'$, $y'' = -Fy' - g$, where F is the resistance divided by the velocity. A qualitative study of the trajectories belonging to this system is made under the sole assumption that $F = F(x', y', x, y, t)$ is a *positive* continuous function of its five arguments. Various inequalities involving range, maximum ordinate, time of flight, etc., are derived, all of these inequalities reducing to equalities in vacuum. A summary of further results obtained under stronger assumptions on F is given; these results are of practical importance in the question of deducing simple approximate formulas for the differential corrections in ballistics.

16. The paper by Major Veblen is a report on experimental work carried out at the Aberdeen Proving Ground, which led to improvements in artillery projectiles. Among the questions investigated were the effects of false ogives, of boat-tailed or stream-lined bases, and of devices for modifying the yaw. Considerable increases in range were obtained by these methods, but the most important gain was obtained by a modification of the rotating band by the removal of some excess copper. This resulted, in the case of the 6-inch gun, in an increase in range of about 2½ miles, and in a diminution of dispersion by a factor of 8. Similar results were obtained in several other guns. It is intended to publish an account

of these studies on rotating bands in an article by Major Veblen and Lieutenant Alger in the *Journal of the United States Artillery*.

These investigations were of an empirical nature, and were carried out, for the most part, without the guidance of a mathematical theory. The absence of a mathematical theory is particularly to be underlined in the case of the rotating band experiments, because semi-official popular accounts of this work have been published in which it is stated that the results were obtained by pure mathematics. The work at the proving ground was done in close collaboration by several men, among whom may be mentioned Lieutenants P. L. Alger, F. E. Fish, and H. Galajikian, and Captain J. W. Alexander. Although the coöperation among all concerned was so close as to make it impossible to trace ideas to their sources, the largest share of the credit must be assigned to Lieutenant Alger, particularly in the rotating band investigations. The results on the 6-inch gun were obtained independently in France by Captain R. H. Kent, after studying the Aberdeen results for the 8-inch howitzer.

17. Professor Birkhoff shows that his earlier work on boundary value and expansion problems for a single ordinary linear differential equation of the nth order (*Transactions*, volume 9 (1908), pages 219–231, 373–395) can be extended to differential systems of the first order. In this manner it is possible to deal with these problems for ordinary differential equations of the nth order in which the parameter does not enter linearly.

18. In a paper by Professor Emch (*Proceedings of the National Academy of Sciences*, volume 4 (1918), pages 218–221) the existence of certain types of closed curves of motion for a particle moving on a sphere in a gravitational field is established. Professor Birkhoff points out in his note that if a point moves on any surface which has a vertical plane of symmetry toward which it is concave, then the existence of such closed curves is an immediate consequence of general results deduced by him earlier (*Transactions*, volume 18 (1917), pages 193–300).

19. Using the methods of Sundman (*Acta Mathematica*,

volume 36 (1912), pages 105–195), Professor Birkhoff proves that if three bodies lie sufficiently near together at $t = 0$ with *assigned* area integral constants not all zero, then at least two of the mutual distances become infinite for $\lim t = \infty$.

20. Professor Kasner shows that the differential equation of the third order defining all the trajectories of a plane field of force admits a first integral of the form $y'' = cF(x, y, y')$ when and only when the field is central (or, as a limiting case, parallel). Generalizations are obtained by imposing some instead of all five of the geometric properties characterizing a dynamical family. (See Princeton Colloquium Lectures, page 10.)

21. In this paper Dr. Ritt discusses the manner in which weighting-factor curves for differential variations behave as the elevation approaches zero. It is found that there exists a limiting weighting-factor curve for each type of atmospheric disturbance. In the case of a range wind the weighting-factor curve is a cubic which depends on the muzzle velocity, but not on the projectile. The weighting-factor curves for a variation in air density and for a variation in the velocity of sound have the same weighting-factor curve for zero elevation, a cubic which is the same for all projectiles and for all velocities. The weighting-factor curves for cross wind approach a parabola which is also independent of the projectile and of the velocity. These limiting curves may be used in the reduction of firings at low angles, when it is desired to avoid short-arc computations. This paper will be offered for publication to the *Journal of the U. S. Artillery*.

22. In transcribing his process of light-signalling Einstein obtained relations connecting the space-time coordinates in two systems, in the form of functional equations. He solved these by passing to partial differential equations and restricted the solutions to linearity by appeal to a homogeneity whose meaning is somewhat obscure. In the present paper, through a somewhat more complete use of functional equations, Professor Lunn shows that the assumption of differentiability is not necessary, and that the needed features of homogeneity are already implicit in the optical and kinematic postulates.

23. In a paper, recently published in the *Annals*,* Dr. Kline gave a non-intuitional definition Σ of sense on closed curves in any space satisfying Professor R. L. Moore's system of axioms Σ_3. It will be remembered that Σ_3, besides being satisfied by an ordinary euclidean plane, is also satisfied by certain spaces which are neither metrical, descriptive, nor separable. In the present paper, Dr. Kline considers the following set of independent postulates for sense on closed curves with respect to their exterior:

Axiom 1. If A and C separate B and D on the closed curve J, then the sense ABC on J is not the same as the sense ADC on J.

Axiom 2. If, on the arc ABC of the closed curve J, E, H and F are points such that on the arc ABC the order $AEHFC$ holds, while EXF is an arc lying except for its endpoints within J, then the sense ABC on J is the same as the sense EHF on $EHFXE$.

Axiom 3. If the sense $A_1B_1C_1$ on J_1 is both the same as the sense $A_2B_2C_2$ on J_2 and as the sense $A_3B_3C_3$ on J_3, then the sense $A_2B_2C_2$ on J_2 is the same as the sense $A_3B_3C_3$ on J_3.

The result obtained is that, if Σ' is any definition of sense on closed curves in a space satisfying Σ_3, then a necessary and sufficient condition that Σ' have the properties of Axioms 1 to 3 is that Σ' be equivalent to Σ.

24. In a class L of Fréchet, a set of points M is said to possess the Heine-Borel-Lebesgue property if, for every infinite family G of subsets of L that covers M, there exists a finite sub-family of G that also covers M. The Heine-Borel-Lebesgue theorem is said to hold true in a given space L if every closed and compact subset of L possesses the Heine-Borel-Lebesgue property. In a recent paper, Fréchet has shown that in order that the Heine-Borel-Lebesgue theorem should hold true in a given class S (i.e., a class L in which the derived set of every set is closed) it is sufficient that that class S should be a class V (a class L in which a distance function exists). He points out however that this sufficient condition is not necessary and raises the question as to what property it is necessary and sufficient that a class S should possess in order that the Heine-Borel-Lebesgue theorem should hold

* Cf. "A definition of sense on closed curves in non-metrical plane analysis situs," *Annals of Mathematics*, 2d series, vol. 19 (1918), pp. 185–200.

true in that class. In the present paper, Professor Moore exhibits one such property. He calls a family G of point sets a monotonic family if, of every two members of G, one of them contains the other one. A monotonic family G is said to be proper if there is no point common to all the members of G. A set of points M is said to have the property K if every monotonic family of closed subsets of M is improper. A necessary and sufficient condition that the Heine-Borel-Lebesgue theorem should hold true in a given class S is that in that class every closed compact point set should have the property K.

If a set of points M is defined as being compact in the new sense in case for every proper monotonic family G of subsets of M there exists at least one point which is a limiting point of every member of G, then the following theorem holds true:

In a class S, in order that a point set M should possess the Heine-Borel-Lebesgue property it is necessary and sufficient that M should be closed and compact in the new sense.

25. In the *Göttinger Nachrichten*, 1910, pages 507–516, F. Bernstein proves that if the class number of the field defined by $e^{2i\pi/p^2}$, where p is prime, is divisible by p but not by p^2, then the relation

$$(px)^p + y^p + z^p = 0$$

is not satisfied in rational integers x, y, z, none zero. In the present paper Mr. Vandiver investigates the divisibility of $\Omega(e^{2i\pi/p^2})$ by p, and shows that Bernstein's assumption is equivalent to the statement that p is a regular prime, and therefore his result is included in Kummer's well known criterion.

26. Professor Beal's paper is in abstract as follows: With one exception, at most, four points of a circle of a congruence generate surfaces normal to lines through them and in planes orthogonal to the circle. If the circle touches the envelope S of its plane, one, two, or three points may coincide at the point of tangency without imposing a condition on the radius R. If S is developable or if the circle is tangent to an asymptotic line, two coincide.

When the center is at the point of tangency these cases arise: With radius $R(u)$ and lines of curvature parametric, two points are on the tangent to $v =$ const. and the normals at them intersect the normal of S at distances $\pm R'(u)\mathcal{E}^{-1/2}$ from the center of normal curvature of $v =$ const.* The other points are symmetrical with respect to the tangent of $v =$ const. and the normals pass through the center of normal curvature of $u =$ const. The last two points are real only when GR'^2 is less than or equal to $\mathcal{E}G + GE + 2DD''$. In the first case the points are distinct and in the second coincide on the tangent to $v =$ const. The above holds true if R is constant. The four points are then on the tangents to the lines of curvature. If S is developable with rectilinear generators $v =$ const. and orthogonal trajectories $u =$ const. three cases exist: $(\partial R/\partial u)^2 < E$, four distinct points; $(\partial R/\partial u)^2 = E$, three coincident points; $(\partial R/\partial u)^2 > E$, two imaginary points.

27. Hurwitz and Silverman have studied transformations of sequences of the type $t_n = \sum_{k=1}^{n} a_{nk}s_k$, where the numbers a_{nk} are defined in terms of certain values of a function $f(z)$ of the complex variable z. Professor Silverman defines a_{nk} by means of a real function $f(x)$ of the real variable x. The condition for regularity of the transformation T_f is obtained simply in terms of $f(x)$ and its continuous derivative $f'(x)$. The condition for the equivalence of two transformations T_f and T_g is that there exist a solution, having certain properties, of an integral equation involving $f(x)$ and $g(x)$. As a special case, the equivalence of Cesàro and Hölder summability of like order, integral or fractional, is deduced. It is further shown that this method in terms of real variables is more general than the one in terms of complex variables.

The results obtained for divergent series are shown to apply also to divergent integrals.

28. The primary object of Mr. Fry's paper is to establish the validity of certain methods of treating differential equations by the use of Fourier's integrals. These methods have been used by a number of mathematicians in the treatment of circuit problems, but the very considerable extent of their field of validity apparently has not been recognized. From

*E, G, D, D'', $\mathcal{E}$ and $\mathcal{G}$ are fundamental quantities for S and its spherical representation.

a standpoint of pure mathematics, the most important contribution of the paper is a definition of the integral

$$\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} dn f(\lambda)\phi(n, t)e^{in(t-\lambda)},$$

which applies even when ϕ does not vanish at ∞. This definition is shown to be consistent with the ordinary Riemann definition, when the latter applies; and the validity of performing addition, subtraction, differentiation and integration under the sign of integration is established. The conception of Stieltjes' integrals is used in carrying out the argument. A minor point of interest concerning the possibility of varying the path of a Fourier integral is touched upon, leading to the reduction of such integrals to closed paths.

29. Riesz has recently derived a large part of the Fredholm theory of integral equations for the equation

$$f(x) = \varphi(x) + \lambda T(\varphi),$$

where $T(\varphi)$ is any completely continuous linear transformation, and in an earlier paper he has proved that every linear transformation, that is linear functional depending on a parameter, can be put in the form

$$T(\varphi) = \int_a^b \varphi(y) d_y K(x, y).$$

After deriving a sufficient condition that a set of discontinuous functions shall be compact, Dr. Fischer has proved that this transformation will be completely continuous if the variation of K in y and its two-dimensional variation are both finite. Another sufficient condition and a necessary condition are also obtained.

30. Professor Emch presents the results of an investigation of the curves described by a spherical pendulum in case that they are closed. The discussion is based upon the parametric representation of the cartesian coordinates of a point of the curve by Weierstrassian σ-functions. It is shown that the closed curves are algebraic, and rational in Greenhill's case, in which the pendulum-bob just reaches the plane of suspension. A classification of these curves is given as well as an

account of their principal geometric properties. The paper was published in the March number of the *Tôhoku Mathematical Journal.*

31. In the theory of conics it is a well known theorem that two inscribed triangles determine a second conic, for which each is self-polar. On a twisted cubic curve a similar theorem was proved by von Staudt and Hurwitz for two tetrahedra. Professor White's paper presents an explicit algebraic formula by which both these theorems are established, and from whose obvious extension others can be inferred. This formula effects in the one case a (2, 2) transformation, in the other a (3, 3) transformation, of the parameter in terms of which points on the locus are rationally expressible. It permits the explicit determination of the infinitely many self-polar triangles of a second conic which are inscribed in the first, as soon as two are known.

32. When the coordinates of a point in space are expressed in terms of three independent parameters u, v, w, the points for which any one of the parameters is constant, say w, lie on a surface. If the parameters are such that the curves $u =$ const. and $v =$ const. on the surface form a conjugate system, we say that space is referred to a triply conjugate system of surfaces. Professor Eisenhart proposes and solves the problem of finding triply conjugate systems for which the three Laplace equations satisfied by the cartesian coordinates have equal invariants. When in particular the triple system of surfaces is orthogonal, the lines of curvature on all the surfaces are isothermic. The problem of finding these orthogonal systems has been solved by Darboux. This paper will appear in the June *Annals.*

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Secretary.