$y = f(x)$. On the positive side of the xy-plane take the curve lying on this cylindrical surface and satisfying the condition $z = P(x)$; on the other side take the curve lying on the cylindrical surface and satisfying the condition $z = -N(x)$. We thus intercept between two curves a part of the cylindrical surface for which x is on (ab). It is not difficult to form a fair intuitive notion of this portion of the surface since $P(x)$ and $N(x)$ are both monotonic non-decreasing. Then the integral (2) is the "area" of the part of this bounded cylindrical surface lying on the positive side of the xy-plane minus the "area" of that part lying on the other side.

UNIVERSITY OF ILLINOIS,
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A DERIVATION OF THE EQUATION OF THE NORMAL PROBABILITY CURVE.

BY PROFESSOR W. D. CAIRNS.

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THE symmetrical distribution of magnitudes about their mean is commonly represented by a "polygon" whose equally spaced ordinates are proportional to the terms of the expansion of $(1 + 1)^n$. The statement is frequently made in textbooks without any proof that as n is increased indefinitely, the equal spaces and the vertical scale being properly controlled, the polygon approaches as its limiting form the normal curve $y = ke^{-h^2x^2}$. The method here given for the proof of this theorem consists essentially in controlling what may be called the points of inflexion of the polygon so that these points approach predetermined positions on each side of the mean. Since an extended and rigorous proof of the probability theorem has been published,* it will suffice to indicate here the general plan of the proof.

* E. L. Dodd, *American Mathematical Monthly*, vol. 20 (1913), p. 128. Since this paper was written, a proof by A. A. Bennett has appeared in this BULLETIN, volume 24, No. 10, page 477. In that proof the area (rather than the standard deviation, as used in the present paper) and the middle ordinate control the curve and Wallis's product formula for $\pi/2$ is used. Since the area and the standard deviation are alike fundamental in the applications to probability, statistics, theory of errors, etc., it would seem that each of these gives a natural method of approach.

By the elementary binomial formula the middle (greatest) term of $(1+1)^{2n}$ is $(2n)!/(n!)^2$; and by the shortened Stirling's formula

$$n! \sim \sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}}$$

this term approximates for large values of n to $2^{2n}/\sqrt{\pi n}$. If then we assign the terms of $\frac{\sqrt{\pi n}}{2^{2n}}(1+1)^{2n}$ to points on the x axis to the left and right of the origin at distances Δx apart, the middle term, assigned to the origin, approximates for large values of n to unity (and thus, by the introduction of a constant factor, to any preassigned value).

The kth term to the right (or left) is

$$\frac{\sqrt{\pi n}}{2^{2n}} \frac{(2n)!}{(n+k)!(n-k)!}$$

and this by Stirling's formula approximates to

$$\frac{\sqrt{n}}{2^{2n}} \frac{(2n)^{2n+\frac{1}{2}}}{\sqrt{2}(n+k)^{n+k+\frac{1}{2}}(n-k)^{n-k+\frac{1}{2}}},$$

i.e., to

$$\left(\frac{n-k}{n+k}\right)^k \bigg/ \left(1-\frac{k^2}{n^2}\right)^{n+\frac{1}{2}},$$

the form here used for our ordinate.

Aside from the factor $\sqrt{\pi n}(2n)!/2^{2n}$, the difference between the kth and the $(k+1)$th term is

$$\frac{2k+1}{(n+k+1)!\,(n-k)!}.$$

Hence the k_σth term for which the ordinates of the polygon decrease most rapidly is characterized by the condition

$$\frac{2k_\sigma - 1}{(n+k_\sigma)!\,(n-k_\sigma+1)!} \leqq \frac{2k_\sigma+1}{(n+k_\sigma+1)!\,(n-k_\sigma)!}$$
$$> \frac{2k_\sigma+3}{(n+k_\sigma+2)!\,(n-k_\sigma-1)!},$$

or by an easy reduction

$$k_\sigma \leqq \sqrt{(n+1)/2} < k_\sigma + 1.$$

If therefore as n increases we choose Δx so that $\sqrt{n/2}\cdot\Delta x\cdot$ equals σ, a predetermined positive constant, the last condition may be written

$$k_\sigma \Delta x \leqq \sigma\sqrt{(n+1)/n} < k_\sigma \Delta x + \Delta x;$$

hence the k_σth term will with increasing n approach the position whose abscissa is σ. If further we set $k\Delta x = x$, i.e.,

$$k = \frac{x}{\sigma}\sqrt{n/2},$$

the kth ordinate, assigned to the position whose abscissa is x, will be

$$\frac{1}{\left(1-\dfrac{x^2}{2\sigma^2 n}\right)^{n+\frac{1}{2}}}\left(\frac{n-\dfrac{x}{\sigma}\sqrt{\dfrac{n}{2}}}{n+\dfrac{x}{\sigma}\sqrt{\dfrac{n}{2}}}\right)^{\frac{x}{\sigma}\sqrt{\frac{n}{2}}}$$

Finally, by evaluating this expression for $n = \infty$, it is found to approach uniformly the limit $e^{-x^2/2\sigma^2}$, whence the equation of the limiting form of curve is $y = e^{-x^2/2\sigma^2}$, the customary form of the normal curve equation save for an arbitrary constant factor.

From the foregoing it also follows that if instead of each ordinate there be used a rectangle of the same height and of base Δx, the sum of the areas of the rectangles approximates asymptotically to $\sqrt{\pi n}\Delta x$, i.e., to $\sigma\sqrt{2\pi}$; in other words, (1) the area under the curve is finite and (2) we have evaluated the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}dx = \sigma\sqrt{2\pi}.$$

The writer has derived a series of formulas, a representative of which is the following:

$$\sum_{k=-n}^{k=n} \frac{(2n)!}{(n+k)!\,(n-k)!}k^2 = \frac{n}{2}2^{2n};$$

i.e., if the binomial coefficients (or terms) of $(1+1)^{2n}$ be multiplied respectively by the squares of the number of the terms counting from the midterm, the sum of the products is as stated. (The theorem may also be stated in terms of mean square deviation.) The use of this formula in the foregoing method gives the result

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}x^2dx = \lim_{n=\infty} \sum_{k=-n}^{k=n} \frac{(2n)!}{(n+k)!(n-k)!}(k\Delta x)^2\Delta x = \sigma^3\sqrt{2\pi}.$$

Similar evaluations are obtained for

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}x^4dx, \quad \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}x^6dx, \text{ etc.}$$

OBERLIN, OHIO.

BÔCHER'S BOUNDARY PROBLEMS FOR DIFFERENTIAL EQUATIONS.

Leçons sur les Méthodes de Sturm dans la Théorie des Équations Différentielles Linéaires et leurs Développements Modernes, professées à la Sorbonne en 1913–1914. Par MAXIME BÔCHER. Recueillies et rédigées par GASTON JULIA. Paris, Gauthier-Villars, 1917. 6 + 118 pp.

IT can be said without fear of contradiction that what may be characterized as the *linear problem* is one of the most central in all mathematics. In algebra this problem concerns itself not only with linear forms and linear equations but also with many phases of the discussion of bilinear and quadratic forms. The results arrived at from an algebraic treatment find immediate application in geometry and mechanics. In the field of analysis the linear differential equation in one or more independent variables has always occupied a position of prime importance and in recent years the study of linear integral equations has not only forged a new and powerful tool but has also exerted a profound influence on the general trend of mathematical thought. The recent development of the theory of linear algebraic equations in an infinite number of unknowns by bridging the gap between the old algebraic field of linear equations and bilinear forms on the one hand and the analytic field of differential equations, integral equations, and bilinear forms in an infinite number of variables on the other, has given a remarkable unity to the various aspects of the general problem. In searching for the theory