MATRICES AND DETERMINOIDS.

Matrices and Determinoids. By C. E. CULLIS. University of Calcutta Readership Lectures. Cambridge University Press, Vol. I, 1913, xii + 430; Vol. II, 1918, xxiv + 555 pp.

THIS treatise was designed to occupy two volumes of theory and one of applications to vector analysis and invariants. The growth of the manuscript however has brought about three volumes for the theory and one for applications. Consequently the present volume instead of closing the theory leaves it still incomplete. The third volume of theory is to include the theory of matrices with functions as coefficients, if we read the indications correctly.

The greater part of the first volume is devoted to the notion of determinoid and theorems connected with this. The matrix itself is studied mainly in connection with the notions of addition, subtraction, and multiplication. A study of the solution of matrix equations of the first degree, which includes systems of linear algebraic equations, also is included in this volume.

The second volume deals with compound matrices, the minors of a matrix, some properties of square matrices, rank of a matrix, transformations of a matrix, equations of the second degree, extravagances of matrices, paratomy and orthotomy of matrices, and three appendices.

As has happened in treatises from some Cambridge mathematicians, a great addition to the existing mathematical vocabulary is to be found in this treatise. Whether so many new terms are necessary of course remains to be seen in their developmental use. One's first impression is, however, that the matter is overdone. Some of them are descriptive enough to explain themselves to some extent, but others are manufactured for the occasion and only to be understood by reference to the text or a glossary. There is a complete and systematic notation throughout, which is highly desirable, and after one has accustomed himself to its method, it is quite intelligible, though successive abbreviation renders it more and more compact.

Definition of Matrix.—The author follows the usual custom and defines a matrix as an assemblage of m rows of n elements each (where m and n are finite for the purposes of this treatise), the elements being numbers in the cases actually considered, and so far as one observes chosen from the real domain usually, but sometimes from the complex domain. The number of rows m is the horizontal order, and the number of columns n is the vertical order. The smaller of these two numbers if unequal, or either if equal, is the efficiency of the matrix, which is thus the order of the maximum determinants that may be formed from the elements of the matrix. Long rows, short rows, leading element, leading line or diagonal, leading vertical row, and leading position, are self-explanatory.

Regarding this definition the reviewer desires to make some comments which apply also to the definitions usually given of vector, tensor, etc. In the first place the term *assemblage of elements* almost invariably leaves out a highly essential phrase: in a certain prescribed order. For instance if the element 2 occurs in any of these entities it has a very different rôle when it is the first element, or the third element, or the second of the third set, etc. This is obvious. Hence the mere assignment of the values of the elements does not define uniquely the entity, whether matrix, vector, or similar entity. If one speaks of the vector (2, 3, 4) he tacitly implies certain notions: namely, what the position of the 2 signifies, the position of the 3, and the 4. The position is actually of more importance than the element in most problems. The definition of a vector as a triple of elements is not sufficient, nor is the definition of a matrix as an m-tuple of n-tuple sets. There seems to be then no valid reason why the qualitative element which designates the rôle of the numerical element should not be put in evidence. For instance, the vector above should be written $2\epsilon_1 + 3\epsilon_2 + 4\epsilon_3$, the ϵ_1, ϵ_2, ϵ_3 being qualitative elements, hypernumbers of a unitary character. Likewise a matrix is expressible completely by the sum $\Sigma a_{ij}\,\lambda_{ij}$, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. The hypernumbers λ indicate the position and the rôle of the coefficients. By thus placing such "assemblages" in the domain of linear algebras, where they properly belong, the whole treatment becomes simple and clear. Particularly the multiplication of matrices is set forth in a much more brilliant light. Further it should be noticed that a matrix of m long rows and n short rows is as much an assemblage of m vectors in an n-dimensional space, or of n vectors in an m-dimensional space, as it is of mn-elements. A matrix may be looked upon as a linear homogeneous substitution, which converts certain variables into others, or it may be considered as a linear vector operator which converts all vectors from the origin into others from the origin, or it may be considered as a dyadic, the sum of several dyads,—all useful definitions and yielding very important results. As a linear vector operator the particular elements entering the matrix are not of so much importance as other features of the matrix. Certain combinations of the elements are invariant, however, in various representations of the linear vector operator, or certain transformations of the matrix considered as an aggregate of elements. These combinations are of great importance. There are other matrices, which have been called covariants of the given matrix, which play a very important part in the theory, when we consider the matrix as an operator.

These considerations lead us to conclude that the mere assemblage of mn numerical values out of whose combinations various forms arise subject to a large variety of theorems is far from being the whole story.

Determinoids.—By determinoid the author means a sum of all the maximum determinants with specified signs that can be formed from the array of elements. Of course if the array is square there is but one such: the determinant of the matrix, as usually understood. If the matrix is rectangular, there must be assigned a rule for the addition of the maximum determinants, and many pages are devoted to the exposition of this rule in various forms, and to various modes of writing out the expansions of the determinoid.

Again the reviewer would remind the reader that out of the mn elements of a matrix an unlimited number of numerical combinations may be formed according to assigned rules, and the only question proper with regard to them is whether they are useful. Of all combinations those which are linear in the elements of each line or column would naturally be suggested as the most useful. Of these the products which are such as to have a number of factors equal to the order of efficiency of the matrix, would take priority. Products of elements chosen one from each of several different lines are called by the author derived products. If the efficiency is r and products of order r are formed in every possible manner from the elements of the matrix, so that no line is represented twice in any one product, the sum of these products with arbitrary coefficients would be one of the useful combinations spoken of above. But the choice of the arbitrary coefficients would have to be governed again by some pragmatic principle. If they are all taken equal to $+1$ then the whole combination has a symmetry easily seen. If half are chosen properly positive and half properly negative, there is a skew symmetry. In each maximum minor we have such skew symmetry. The author was evidently aiming at some such result in his choice of coefficients for the derived products in forming his determinoid. But an examination of the theorems relating to determinoids, which he seems to think will be useful for the applications, makes it evident that to a large extent the coefficients of the maximum minors could just as well have been arbitrary and the theorems would still hold. We must then feel rather doubtful as to the utility of the determinoid when it is not a simple determinant.

To understand just what happens we may have recourse to the qualitative units. The elements of the row may be looked upon as defining a vector α_s. For simplicity we will suppose that the efficiency is m. Then if we set up a multiplication of these vectors by means of an alternating multiplication of the qualitative units $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ (as is often done in Scott's Determinants, for instance), this will be an $\frac{n!}{m!\,(n-m)!}$ vector, of class m in a space of n dimensions, whose coordinates (coefficients) are the maximum minors of the matrix. As an alternating expression this product has certain properties and these are the properties which are to be found in the determinoid. For instance the addition of a long row to another is equivalent to adding one of the vectors to another, which will not affect the product. The determinoid itself is the scalar product of this alternating product and another vector of the same order, whose coefficients are $+1$ and -1 chosen according to the rules previouly laid down. Any other vector of this order might in most cases as well have been chosen. We may reach the reciprocal matrix also with these alternating products, for if we construct a vector β_j of order 1, linear in the arbitrary vectors $\lambda_1, \lambda_2, \ldots, \lambda_m$, any chosen set of m linearly independent vectors, the coefficient of λ_i being the scalar product of the alternating product of $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_m$ and that of $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m$, where j runs from 1 to m, and signs follow a determinant rule, then these vectors $\beta_1, \ldots, \beta_j, \ldots, \beta_m$, define the reciprocal matrix. In the case actually chosen the vector which is used to give the determinoid is also chosen as a sum of alternating products of m out of n vectors. Any arbitrary vector of the same order would do. In the case of a square matrix of course these all become multiples of a single unitary one, and the reciprocal of the square matrix which is not singular is therefore unique. Professor Cullis gets a unique reciprocal in the rectangular case only because of his special choice of the vector which gives the determinoid. By leaving this choice open for different problems he would have arrived at a much more flexible system. The product of a matrix and its reciprocal is of course the mth power of its determinoid. This would be true for any choice of the arbitrary vector.

Products of Matrices.—The elements of a product are formed by combining corresponding elements from the rows of the prefactor and the columns of the postfactor. These are the active rows of each respectively. The vertical rows of the prefactor and the horizontal rows of the postfactor are the passive rows. The passivity of either factor is the number of passive rows it contains. This is evidently a function of its place as prefactor or as postfactor. When the passivities are not equal they are made so by the adjunction of lines of zeros to the matrix with the smaller passivity. The product will then have the same number of horizontal rows as the prefactor, and vertical rows as the postfactor. The product of two vectors usually called their scalar product, can now be defined as the determinant (or determinoid equally in this case), of the product of two matrices, the first with one row, the second with one column. We may instead of inserting lines of zeros, strike out the redundant horizontal rows of the second or the redundant vertical rows of the first, and reach the same product. The latter method is preferable since it makes evident at once the rule for the cancellation of factors in a product which vanishes. This rule is stated for the most general case in these terms: The equation $AXB = 0$ leads to $X = 0$ as a necessary consequence, when and only when the ranks of A and B are equal to their passivities in the given product AXB. From this we have that $AX = AY$ leads to $X = Y$, as a necessary consequence, when and only when the rank of A equals its passivity in AX and in AY. And finally $AXB =$

AYB leads to $X = Y$, as a necessary consequence, when and only when the ranks of A and B equal their passivities in the two products. On the basis of these theorems we can proceed to solve linear equations in matrices or sets of linear equations in n variables, for these are reduced to linear matrix equations. For instance the system

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad (i = 1, \ldots, m),$$

becomes the matrix equation

$$|a_{ij}||x_j| = |b_i|, \qquad (i = 1, \ldots, m; j = 1, \ldots, n).$$

Compound Matrices.—These are defined and treated in the second volume. The notations are generalizations of those of the preceding volume. A compartite matrix is one whose elements all vanish except those belonging to a number of mutually complementary minors. By interchanging rows and columns it can always be brought to a form in which there are minors (usually rectangular) which form a diagonal set, all others being zero. This is called the standard form. The rank of a compartite matrix is the sum of the ranks of its component parts. The conjugate reciprocals of certain compound matrices are considered, the results being of use later. The primaries of the minor determinant Δ, of order r, in a matrix of orders m, n are the minors of order r which differ from Δ in only one row, the horizontal primaries differing in one horizontal row, the vertical in only one vertical row. The primary superdeterminants of a minor Δ of order r are all the minors of order r plus one which contain Δ as a minor. There are several theorems on the possible ranks of a matrix containing a given minor.

Relations between the Elements and the Minors.—In this chapter a quite considerable number of identities will be found, deduced for the most part by a skilful use of a matrix and its conjugate reciprocal. Many of them are identities well known in generalized vector analysis. In fact if we consider that the matrix of n short rows and m long rows is an aggregate of n vectors in an m-dimensional space, we can at once derive many of the formulas. Sylvester's identities satisfied by those primary superdeterminants of one determinant Δ which lie in another determinant D containing Δ are included in the list. There is also a consideration of equivalent matrices, that is, matrices A, B such that $A = BC$ or $A = CB$, where C is an undegenerate matrix. These are used in the theorem that two systems of r unconnected linear equations on n variables which both have finite solutions are mutually equivalent when and only when the related matrices of coefficients are equivalent.

Square Matrices.—A large part of this long chapter is devoted to the properties of the square matrices whose elements are the minors of order r of a given matrix. We find such terms introduced as, co-joint matrices: each consisting of the minors which are complementary to the minors that constitute the other; corresponding and anti-corresponding minors: corresponding minors being formed in the same manner from corresponding elements of co-joint determinants, anti-corresponding minors being each the cofactor in one of two co-joint matrices to the minor corresponding to a given minor in the other. Several theorems are given relating to these matrices. Matrices that are formed by bordering a given matrix are studied. Reciprocal matrices give several theorems. Symmetric matrices and skew-symmetric matrices have some pages each. However, the whole subject of the characteristic equation of the matrix, the general equation, the scalar invariants, the related matrices, which the reviewer has called the chi functions of the matrix, are not even mentioned. Indeed after going over some 600 pages of the author's treatise, with the expectation that somewhere the matrix as an operator, not necessarily a linear substitution, but only an operator on other matrices, will be treated, one finds with disappointment that apparently this phase of the subject is not to be found in the author's scheme of development. The reduction of a matrix to its canonical form, the subject of elementary divisors, and all related topics are not mentioned. Perhaps the succeeding parts of the treatise will remedy this serious defect, but no hint that this will happen is given. The nearest approach seems to be in Chapter XVI, which deals with the equigradient transformations of a matrix with constants for elements.

Equigradient Transformations of a Matrix.—A transformation of this kind is equivalent to the transformation linearly of the variables in a bilinear form corresponding to the matrix. The greater part of the chapter is devoted to a consideration of the various transformations that will reduce the matrix to a matrix consisting of square matrices down the main diagonal, other elements being zero. This corresponds of course to the reduction of the corresponding system of linear equations to a system which is reduced. In particular, symmetric matrices reduced to symmetric matrices are studied. The signants of the matrix are defined, and the matrix is called indefinite when there is both a positive signant and a negative signant, otherwise definite. The invariants of such transformations are not mentioned, nor is the significance of the transformation made clear.

Matrix Equations of the Second Degree.—The types considered are of course very special. We find equations of the unsymmetric forms

$$XY = AB, \ XY = C.$$

All others considered are symmetric, such as, for instance, $X'X = I$, where X' is the transverse (conjugate) of X, and I is the identity matrix; $X'X = A'A$, $X'X = C$, $X'AX = C$. The case $X'X = I$ evidently gives the orthogonal, unitary matrices. This is solved in full in the sense that methods are given by which any number of special solutions may be constructed. An application is given to the rotations of a rigid body in three-dimensional space.

The Extravagances of Matrices.—The degeneracy of a matrix of rank r is the number by which its rank falls short of its efficiency, that is the number of long rows it contains. Or in other words, if a matrix is given by a set of vectors in a space of m dimensions (called a spacelet in the text) and its rank r is less than m, $m - r$ is the degeneracy of the matrix. The extravagance of an undegenerate matrix of rank r is the degeneracy of the self-transverse product of the matrix and its transverse. The extravagance is never negative nor more than the rank, nor more than the order minus the rank. An extravagant matrix is such that the sum of the squares of all its maximum minors is zero, that is to say, if we consider that the matrix is given by m vectors of an n-dimensional space, and form the alternant $A_m\alpha_1\alpha_2 \ . \ . \ . \ \alpha_m$, the tensor or absolute magnitude of this vector is zero. A real undegenerate matrix evidently has zero extravagance. A completely extravagant matrix has extravagance equal to its rank. A matrix is plenarily extravagant when its extravagance is equal to the difference of its orders. Matrices are mutually orthogonal when the product of one into the transverse of the other is zero, and mutually normal, when mutually orthogonal and also with ranks whose sum is n, the number of short columns in each. These terms lead to several theorems, and these to theorems as to spacelets. In a space of n dimensions these notions are connected with the absolute quadric.

Paratomy and Orthotomy.—A matrix of rank r may be considered to be an assemblage of vectors in a space of order r, called a spacelet, and if two spacelets have a spacelet in common as their intersection, then its rank is the mutual paratomy of the two. The mutual orthotomy is the degeneracy of the product of one matrix into the transverse of the other. There are several theorems involving these numbers, and these have evident applications to spaces of different dimensions.

The reviewer has endeavored to give as briefly as possible the main features of the treatise so far as it has been published, although the very large number of theorems and developments make this a difficult thing to accomplish. There remain to be added only a few general comments. The treatment is complete in the sense that it appeals to no other discipline for its methods or its proofs. Where vectors might have been used as operands they are always one-rowed matrices. One might question whether the same theorems might not be reached by other methods in less space and with much more direct connection with the field of applications. The reviewer believes that a judicious use of a generalized vector analysis (such as has been reported on in other places) would assist very much. However the author of a treatise must be permitted to develop it along his own lines and the really proper question is whether he succeeds in doing what he sets out to do, as he desires to do it. Of this there seems little doubt in considering the present treatise. If the remainder of the work is as full of useful theorems and applications, the completed treatise will remain for a long time a valuable place of reference. It is to be hoped that a complete bibliography will be added to the final volume, as well as a glossary. There are omissions which may be supplied in later volumes. For instance no work on matrices can be complete if it leaves out the consideration of the invariant regions, the projective regions, the shear regions of the matrix, the elementary divisors, the scalar and vector and matrix invariants, the connection with dyadics and linear substitutions, the related triadics, polyadics, etc. This is a very extended field with numerous ramifications, and a complete treatise is in duty bound to consider these. On the side of determinants, which is the really major part of the treatise, one might insist on a consideration of the numerous special forms of determinants. The related matrices have interesting properties. Then finally the whole subject of groups of matrices leading to the field of linear associative algebra needs consideration, as well as the modern developments in the study of infinite matrices. There does not seem to be any indication that these are to be treated at all.

The treatment is quite detailed, with numerous numerical examples, rather loose in its development, and lacking in synthesis, so that the reader becomes bewildered with the multitudinous formulas and other details. A synopsis of it would be useful. There are some errors easily noticed

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NOTES.

THE seventy-second meeting of the American association for the advancement of science was held at St. Louis, December 29 to January 3, under the presidency of Dr. SIMON FLEXNER. Professor O. D. KELLOGG was vice-president and Professor F. R. MOULTON secretary of Section A. The address of Professor G. D. BIRKHOFF, as retiring vice-president of Section A, on "Recent advances in dynamics," was delivered on December 30. This address was published in *Science* of January 16. Professor D. R. CURTISS was elected vice-president of Section A for the next two years. Among the societies meeting at St. Louis in affiliation with the Association were the Chicago and Southwestern Sections of the American Mathematical Society and the Missouri Section of the Mathematical Association of America.

THE fifth annual meeting of the Mathematical Association of America was held at Columbia University, New York City, on Thursday and Friday, January 1–2, immediately following the annual meeting of the American Mathematical Society.