THE TWENTY-SIXTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-sixth annual meeting of the Society was held at Columbia University on Tuesday and Wednesday, December 30–31, extending through two sessions on each day. The attendance included the following ninety-six members:

Dr. J. W. Alexander, Mr. P. L. Alger, Professor R. C. Archibald, Professor C. S. Atchison, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor E. G. Bill, Professor C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Professor Daniel Buchanan, Professor R. W. Burgess, Professor W. D. Cairns, Miss M. F. Chadbourne, Dr. Teresa Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor C. H. Currier, Dr. Mary F. Curtis, Dr. Tobias Dantzig, Professor J. V. DePorte, Professor C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor T. M. Focke, Professor C. H. Forsyth, Mr. T. C. Fry, Professor A. S. Gale, Professor W. V. N. Garretson, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor H. E. Hawkes, Professor Olive C. Hazlett, Professor L. A. Howland, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. E. A. T. Kircher, Professor P. A. Lambert, Dr. K. W. Lamson, Professor D. D. Leib, Professor Florence P. Lewis, Professor P. H. Linehan, Professor Joseph Lipka, Professor C. R. MacInnes, Professor H. H. Mitchell, Professor C. N. Moore, Professor Frank Morley, Dr. H. C. M. Morse, Professor G. W. Mullins, Mr. F. H. Murray, Professor F. W. Owens, Dr. Helen B. Owens, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Arthur Ranum, Professor H. W. Reddick, Dr. C. N. Reynolds, Jr., Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. W. G. Simon, Professor C. G. Simpson, Professor Mary E. Sinclair, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Mr. J. J. Tanzola, Dr. J. S. Taylor, Mr. H. M. Terrill, Professor H. D. Thompson, Mr.

H. S. Vandiver, Professor Oswald Veblen, Mr. J. L. Walsh, Mr. H. E. Webb, Dr. Eula A. Weeks, Professor Mary E. Wells, Professor H. S. White, Professor E. E. Whitford, Dr. Norbert Wiener, Professor F. B. Wiley, Professor Ruth G. Wood, Professor J. W. Young.

President Frank Morley occupied the chair, being relieved at the last session by Professor J. L. Coolidge. The Council announced the election of the following persons to membership in the Society: Dr. H. E. Bray, Rice Institute; Professor I. L. Miller, Carthage College; Dr. Helen B. Owens, Cornell University; Professor E. W. Pehrson, University of Utah. Ten applications for membership in the Society were received.

The following resolutions, introduced by Professor R. C. Archibald as chairman of the committee on bibliography, were adopted by the Council:

1. The Council regards the preparation and publication, in America, of a dictionary of mathematical terms as not only most desirable but also entirely feasible, provided that financial aid for the preparation of the manuscript can be secured.

2. Impressed with possibilities for the more extensive development of pure and applied mathematics in America, and with the importance of such development to the nation, the Council records its conviction that there are undertakings whose active consideration would be highly desirable if adequate financial assistance might be regarded as available. Among such undertakings are: 1. The preparation and publication by societies or individuals of surveys, introductory monographs, translations, memoirs, and treatises, in important fields, including the history of mathematics. 2. The organization of research fellowships. 3. The preparation and publication of an encyclopaedia of mathematics in English. 4. The preparation and publication of an annual critical survey, in English, of the mathematical literature of the world. 5. The preparation and publication of a biographical and bibliographical dictionary of mathematicians.

It was decided to proceed with the incorporation of the Society under the general law of the State of New York. A committee was appointed to consider plans for the organization and administration of the Society after the retirement of the present Secretary and Librarian from their offices at the close of the present year. A committee was also appointed to consider the formation of an international union of mathematicians. The committee on mathematical requirements presented a report, which was laid over for consideration at the February meeting.

The total membership of the Society is now 733, including 80 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 393; the number of papers read was 187. The number of members attending at least one meeting during the year was 252. At the annual election 156 votes were cast. The Treasurer's report shows a balance of $10,692.23, including the life-membership fund of $7,168.87. Sales of the Society's publications during the year amounted to $1,811.52. The Library now contains 5,690 volumes, excluding some 500 unbound dissertations.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor C. N. HASKINS, Professor R. G. D. RICHARDSON.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1922,

Professor T. H. HILDEBRANDT,	Professor W. A. MANNING,
Professor EDWARD KASNER,	Professor H. H. MITCHELL.

The meeting of the Society immediately preceded that of the Mathematical Association of America on January 1–2. A very pleasant occasion was the joint dinner of the two organizations on New Year's eve with an attendance of 114 members and friends.

The following papers were read at the annual meeting:

(1) Dr. H. F. MACNEISH: "The sum of the face angles of a polyhedron in space of n dimensions."

(2) Dr. G. A. Pfeiffer: "A connected set of points which contains no continuous arc."

(3) Professor C. J. Keyser: "Fundamental types of groups of relations of an infinite field."

(4) Professor Joseph Lipka: "The theorem of Thomson and Tait and its converse in space of n dimensions."

(5) Professor A. A. Bennett: "Poncelet polygons in higher space."

(6) Professor A. A. Bennett: "Continuous matrices, algebraic correspondences, and closure."

(7) Professor L. A. Howland: "Concerning points of inflection on a rational plane quartic."

(8) Dr. H. C. M. Morse: "Geodesics motion on a surface of negative curvature."

(9) Professor J. L. Coolidge: "The geometry of hermitian forms."

(10) Professors H. B. Phillips and C. L. E. Moore: "Rotations in space of even dimensions."

(11) Professors C. L. E. Moore and H. B. Phillips: "Note on geometric products."

(12) Professor O. E. Glenn: "A memoir upon formal invariancy with regard to binary modular transformations."

(13) Professor O. E. Glenn: "The invariant problem of the relativity transformations of Lorentz appertaining to the mutual attraction of two material points" (preliminary report).

(14) Dr. Norbert Wiener: "The mean of a functional of arbitrary elements."

(15) Dr. Norbert Wiener: "Bilinear operations generating all operations rational in a domain Ω."

(16) Dr. Norbert Wiener: "Fréchet's calcul fonctionnel and analysis situs."

(17) Dr. Norbert Wiener: "A set of postulates for fields."

(18) Mr. J. L. Walsh: "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function."

(19) Mr. J. L. Walsh: "On the proof of Cauchy's integral ormula by means of Green's formula."

(20) Professor Dunham Jackson: "On the order of magnitude of the coefficients in trigonometric interpolation."

(21) Mr. P. L. Alger: "A problem of electrical engineering."

(22) Professor W. B. FITE: "Properties of the solutions of certain functional differential equations."

(23) Professor W. C. GRAUSTEIN: "Determination of the pairs of ordered real points representing a complex point."

(24) Dr. J. S. TAYLOR: "Sheffer's set of five postulates for Boolean algebras in terms of the operation 'rejection' made completely independent."

(25) Professor C. H. FORSYTH: "${}_nE_x$, the magic wand of actuarial theory."

(26) Professor C. H. FORSYTH: "A formula for determining the mode of a frequency distribution."

(27) Professor DANIEL BUCHANAN: "Asymptotic orbits near the equilateral triangle equilibrium points in the problem of three finite bodies."

(28) Professor C. L. BOUTON: "The definition of birational transformations by means of differential equations."

(29) Professor W. C. GRAUSTEIN: "Area-preserving, parallel maps in relation to translation surfaces."

(30) Dr. C. N. REYNOLDS, Jr.: "Note on linear differential equations of the fourth order whose solutions satisfy a homogeneous quadratic identity."

(31) Professor J. E. ROWE: "A practical problem of aerodynamics and thermodynamics."

(32) Professor W. B. CARVER and Mrs. E. F. KING: "A property of permutation groups analogous to multiple transitivity."

(33) Professor OLIVE C. HAZLETT: "Some pseudo-finiteness theorems in the general theory of modular covariants."

(34) Dr. MARY F. CURTIS: "Note on the rectifiability of a twisted cubic."

(35) Dr. TERESA COHEN: "The representation of fractions of periods on algebraic curves by means of virtual point sets."

(36) Professor C. A. FISCHER: "Necessary and sufficient conditions that a linear transformation be completely continuous."

(37) Dr. S. D. ZELDIN: "On the structure of finite continuous groups with a single exceptional infinitesimal transformation."

(38) Mr. J. L. WALSH: "On the location of the roots of the derivative of a polynomial."

Dr. Zeldin was introduced by Professor Lipka. The paper of Dr. MacNeish, the first paper of Professor Bennett, the papers of Professors Phillips and Moore, the third and fourth papers of Dr. Wiener, the first paper of Professor Forsyth, and the paper of Professor Carver and Mrs. King were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. MacNeish obtains the following formula:

$$\Sigma = \left[\tfrac{2}{3}n(n-1)a_0 - \frac{8}{n-2}a_3 \right]$$

right angles for the sum of the face angles of the plane faces of a simple polyhedron in n-space, where a_r $(r = 0, 1, \ldots, n-1)$ represents the number of r-space faces. A simple polyhedron in n-space is defined as a polyhedron in which ${}_{n-i}C_{r-i}$ of its r-space elements meet at each i-space $(r > i;$ $i = 0, 1, \ldots, n-2;$ $r = 1, 2, \ldots, n-1)$. If more than ${}_{n-i}C_{r-i}$ of the r-space elements meet at any i-space, the polyhedron is called a multiple polyhedron. Formulas are developed for the sum of the face angles of certain types of multiple polyhedrons.

2. In this paper Dr. Pfeiffer presents an example of a connected set of points which contains no continuous arc. This example is the set of points (x, y) such that

$$y = \sum_1^\infty c_n \phi(x - \omega_n) \qquad (-\tfrac{1}{2} < x < \tfrac{1}{2}),$$

where

$$\phi(x - \omega_n) = \sin \frac{1}{x - \omega_n},$$

$x \neq \omega_n$ and $-1 - \omega_n < x < 1 - \omega_n$, $\phi(0) = 0$; ω_n is the abscissa of the nth rational point of the interval $-\tfrac{1}{2} < x < \tfrac{1}{2}$ with respect to a definite enumeration of the rational points of that interval and $c_1, c_2, c_3, \ldots$ is a denumerable set of positive numbers such that $\sum_1^\infty c_n$ is convergent.

3. In Professor Keyser's paper the relations of an infinite field (coincident with the domain and the codomain) are

viewed as falling under nine types: $1-1$, $1-s$, $s-1$, $s-s$, $1-m$, $m-1$, $s-m$, $m-s$, $m-m$, where m means many (i. e., two or more) and s means some in the sense that a relation whose type symbol involves s contains at least one couple in which s is represented by only one term and at least one couple in which s is represented by more than one term. A class of relations containing one or more relations of each of k of the nine types, as $t_1, t_2, \cdots, t_k$, is said to be of the type $t_1 \smile t_2 \smile \cdots \smile t_k$. There thus arise 511 class types, called fundamental types. Among the infinitude of classes of a given type there is one class which includes every subclass of the same type. This distinguished class is said to be a fundamental class. Accordingly there are 511 fundamental classes, one for each fundamental type. Given a rule by which any two relations of the field can be combined so as to yield a relation of the field, there arise two problems: (1) To determine which of the fundamental classes are groups under the rule; (2) To determine which of the fundamental classes contain group subclasses of the same type as that of the containing class. The solution of (1), the rule being that of relative multiplication, was given by the author in a paper presented at the last April meeting of the Society. The present paper gives the solution of (2) for the same rule, and the solutions of both problems where the rule of combination is that of logical addition of relations.

4. The theorem of Thomson and Tait for space of n dimensions may be stated as follows: if from all points of an arbitrary hypersurface (space of $n-1$ dimensions) particles not mutually influencing one another be projected normally in a conservative field of force, points which they reach with equal actions lie on a hypersurface cutting the paths at right angles. The paths form a system of $\infty^{2(n-1)}$ trajectories for each value of the constant of energy. The converse theorem proved by Professor Lipka states: if a system of $\infty^{2(n-1)}$ trajectories in space of n dimensions is such that any ∞^{n-1} trajectories of the system which meet an arbitrary hypersurface (space of $n-1$ dimensions) orthogonally admit of ∞^1 orthogonal hypersurfaces, then the system may be considered as the trajectories in a conservative field of force. The purely geometric part of the theorem is true for all *natural* families of

curves. The n-dimensional spaces considered have constant curvature.*

5. Professor Bennett's first paper appears in full in the present number of the BULLETIN.

6. This paper by Professor Bennett shows how an algebraic correspondence may be viewed as a matrix with a continuous array of elements. The result of two successive correspondences is represented by the matrix obtained with the usual rules for a product. The sum of two matrices has an analogous interpretation in correspondences. The matrix representation emphasizes a weighting of correspondences which is suggested, but less readily, by geometric considerations. The meaning of the inverse of reducible correspondences becomes precise only when weights are assigned in the manner required by the matrix representations. The inverse of a weighted algebraic correspondence may fail to exist altogether or may not be algebraic when these terms are carefully defined. The question of the algebraic character of an inverse that exists is entirely equivalent to the closure of the original correspondence, and to the existence of variable closed algebraic configurations on the fundamental manifold.

7. It is well known that the six inflexions of a rational plane quartic lie on a conic. To the best knowledge of the author, however, there is no proof of this theorem which is at once complete and also based on elementary considerations. Professor Howland's paper attempts such a proof, and follows that by special consideration of the cases of a quartic with a triple point.

8. The study of dynamical systems leads at once to the study of geodesics as a type of motion. A primary purpose of Dr. Morse's study of geodesics was to establish the existence of a class of recurrent motions called discontinuous recurrent motions by Professor Birkhoff, and to examine a case where, if the recurrent motions, not simply periodic, existed at all, they would furnish a very general example of such motions. The surfaces of negative curvature considered are surfaces

* For a discussion of these theorems for space of three dimensions see Edward Kasner, *Trans. Amer. Math. Society*, vol. 11 (1910), pp. 121–140.

which may be cut up into n simply connected portions (n arbitrary, and exceeding one) none of which lie wholly in any finite portion of space, together with one portion S which lies wholly in a finite portion of space, and which may be of arbitrary connectivity exceeding two. Geodesics are considered which if continued indefinitely in either sense lie wholly on a finite portion of the surface. A fundamental result of this paper is a one-to-one representation of these geodesics in terms of the $2p$ fundamental contours of S, together with the n closed curves bounding S and separating S from the remaining portions of the original surface. A second paper will apply the results of this paper to the theory of dynamical systems.

9. In Professor Coolidge's paper the problem of reducing a hermitian form of non-vanishing discriminant is discussed. Special attention is paid to those collineations of n-space which leave invariant the form $\Sigma x_i \bar{x}_i$ ($i = 0, 1, 2, \cdots, n$).

10. Professors Phillips and Moore treat rotations in space of $2n$ dimensions by means of the Gibbs dyadic, deriving in particular a form

$$\phi = e^{I \cdot M} = e^{I \cdot \Sigma q_i M_i}$$

for such a rotation. Here M is a complex two-vector, M_i a set of completely perpendicular plane vectors, and q_i the angles of rotation in those planes, I being the identical dyadic and $I \cdot M$ the dot product of Lewis. A sufficient condition that two rotations ϕ and ϕ' be commutative is that the vector of the dyadic $(I \cdot M)$ $(I \cdot M')$ be zero. This is also necessary unless two or more of the q's equal $\pm \pi$.

In four dimensions any rotation can be expressed as the product of two equiangular rotations

$$e^{qI \cdot (M_1 + M_2)} \quad \text{and} \quad e^{q'I \cdot (M_1 - M_2)}.$$

These equiangular rotations have the property of rotating all vectors through the same angle and are analogous to the rotations resulting from the multiplication of quaternions referred in the first case to right-hand axes, in the second to left-hand ones.

11. Products which in terms of units have identities of the form $k_{12}k_{34} = k_{1234}$ (outer product of Grassmann) and $k_1 k_{12} = k_2$

(inner product) are well known. Professors Moore and Phillips consider such products as $k_{12}k_{13} = k_{23}$, this particular one being an extension of Gibbs' cross product. These occur in the analytic expression of complete perpendicularity and in the study of commutative rotations.

12. The memoir of Professor Glenn comprises a general exposition relating to the following subdivisions: Invariants belonging to prescribed domains of transformations of finite order on n variables. A complete system under the above theory, in the $G\ F(p^2)$, of a binary quantic of order m. Systems of universal covariants of the groups G_x: $x_1 \equiv x_1' + x_2'$, $x_2 \equiv x_2' \pmod{p}$, H_x: $x_1 \equiv x_1' + x_2' + x_3'$, $x_3 \equiv x_3' \pmod{p}$, and of the simultaneous groups G_x, G_y. A theorem upon modular systems according to which the formal concomitants modulo p of a binary m-ic are constructed as simultaneous systems of sets of covariants of lower order. Development of a fundamental system of covariants for a linear form and a quadratic modulo 2, of a single binary quartic modulo 2, and of a complete system of semivariants of the quartic modulo 3. Somewhat extensive data concerning a complete system of covariants of the quartic modulo 3 are also included. The last section of the paper gives a theory of complete systems under the transformations on the velocities and accelerations which figure in the theory of relativity of motion.

13. The components F_x, F_y, F_z in the theory of the mutual attractions of two material points are invariantive under the linear transformations S on four variables, called by Lorentz the general transformations of relativity (time being a fourth dimension). A theory of concomitants in the domain of the coefficients of S, based upon the equations of the four planes upon its poles, is treated in Professor Glenn's paper.

14. Dr. Wiener shows how, by applying to E. H. Moore's notion of development the notion of a weighting (i.e., of a measure of volume), a type of mean can be obtained which, under certain very general hypotheses, will be an example of the generalized integration of P. J. Daniell. The means of a function of n variables and of a function of a discrete infinity of variables are shown to be examples of this concept, while it is proved that by a proper "weighting" the mean or

integral of a function of a line can be defined. In this connection, the concept of the "coefficient of irregularity" of a function is introduced, viz.,

$$\operatorname{Max}\frac{\Delta f(x)}{\Delta(x)},$$

and it is proved that a bounded set of functions whose coefficients of irregularity are all less than some finite quantity is extremal in the sense of Fréchet.

15. An operation $f(x_1, x_2, \ldots, x_p)$ is said to generate a class K of operations when

(1) K contains f;

(2) if K contains $g(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k)$, it also contains $g(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k)$ and $g(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k)$.

(3) if K contains $g(x_1, x_2, \ldots, x_k)$ and $h(x_1', x_2', \ldots, x_l')$, K also contains the operation on $x_1, x_2, \ldots, x_{i-1}, x_{i+1} \ldots, x_k, x_1', x_2', \ldots, x_l'$ formed by substituting h for x_i in g.

(4) K contains no proper subset with the three preceding properties.

Dr. Wiener proves that the operations that are bilinear and generate the set of all rational operations with rational coefficients may be reduced by a rational linear transformation to one of the forms

$$\frac{x-y}{x+Gy} \quad (G \text{ rational and } \neq -1)$$

$$\frac{n(x-y+xy)}{ny+xy} \quad (n \text{ integral and } \neq 0),$$

and that every such operation will generate all rational operations with rational coefficients. The results are extended in part to other domains of rationality.

16. Dr. Wiener defines continuous deformations and their limits on the basis of the limit of a sequence of points. The notion of a one-parameter family of deformations follows readily, and an n-space can be defined by employing this concept together with a process of mathematical induction.

Certain properties of the n-spaces selected from a given space are discussed, and finally a categorical set of postulates is given for the analysis situs of a space of any finite number of dimensions.

17. Dr. Wiener shows that all algebraic operations are obtainable by the iteration of $x|y = 1 - x/y$. A set of postulates is derived for this operation. This set is consistent, independent, and equivalent to the sets given by E. V. Huntington for fields. It is also shown that every operation and concept in complex algebra can be derived from the operation $1 - K(x/y)$, where K stands for "conjugate of."

18. Mr. Walsh's paper is a continuation of a recent paper published in the *Transactions* under the same title. It is proved as a lemma that if three variable points of the complex plane determine a fourth variable point by a real constant cross ratio with those three, and if the three points have as their respective envelopes closed regions bounded by circles, then the envelope of the fourth point is a fourth closed region bounded by a circle. This lemma is applied in proving that if three circular regions contain respectively k roots of a binary form of degree p_1, the remaining $p_1 - k$ roots of this form, and all the roots of a second form of degree p_2, then these three circular regions and the fourth circular region of the lemma corresponding to the cross ratio p_1/k contain all the roots of the jacobian of the two forms. If the four regions have no common point, they contain respectively the following numbers of roots of the jacobian: $k - 1$, $p_1 - k - 1$, $p_2 - 1$, and 1.

19. Mr. Walsh's second paper appeared in full in the January BULLETIN.

20. It is well known that there is a remarkable and far-reaching parallelism between the properties of the trigonometric interpolating formula determined by the values of a given function at a set of equally spaced points, and those of the formula of approximation obtained by taking the first terms of the Fourier's series for the same function. This parallelism is closer than would be indicated at first sight by the correspondence of sums and integrals in the formulas for the coefficient of the two approximating functions. It has

to be regarded as due in part to the special properties of the trigonometric functions, not merely to the definition of an integral as the limit of a sum. Professor Jackson's paper is concerned with the order of magnitude of the coefficients in the interpolating formula, in case the given function is of limited variation, or has a derivative of limited variation. The method of partial summation is used to give results analogous to those which Picard (Traité d'Analyse, volume 1) obtains for the Fourier's series by the second law of the mean and integration by parts. Incidentally, in the case of a derivative of limited variation, a relatively simple proof is obtained for the convergence of the Fourier's series, as well as of the interpolating formula, to the value of the given function.

21. Mr. Alger presents an unsolved problem of electrical engineering which is of considerable practical interest to engineers. The problem is to determine the distribution of magnetic flux and current density across the thickness of an infinite plane lamina of iron, when an harmonically varying magnetizing force is applied. Under the assumption of uniform permeability, the problem has been completely solved, and its solution may be found in any standard advanced text book of electrical engineering. The solution is in this case analogous to that of the flow of heat in a plane, and the flux and current densities at any point are proportional to the hyperbolic functions of a complex quantity. In actual practice, however, the permeability varies over a wide range and is directly dependent on the magnetizing force. The solution of the problem, when the permeability is assumed variable, has never been obtained, and the whole problem under this assumption is so little understood that the simplest qualitative results of a mathematical study of it should prove valuable. When the permeability is variable, the harmonic impressed magnetizing force sets up magnetic fluxes of all odd multiples of the impressed frequency, and the solution of the problem involves the determination of the coefficients in a Fourier's series. The law of variation of permeability is a purely empirical one, so that one difficulty in the problem is the choice of a law of variation that will make a solution possible while it still sufficiently represents the facts.

22. In this paper Professor Fite discusses the oscillatory properties of the solutions of binary functional differential equations of the form $y^{(n)}(x) + r(x)y(k - x) = 0$, where $|r(x)| > b > 0$ for all finite values of x. He shows that when n is odd every solution changes sign an infinite number of times, but that when n is even a solution may change sign only a finite number of times. Whether this finite number is odd or even depends upon the sign of $r(x)$. He also determines certain necessary conditions in order that these equations, for $n \leqq 2$, may have odd or even solutions.

23. A complex point P in a complex space of n dimensions, since it depends on $2n$ real parameters, may be represented by a pair of real points P_1, P_2 in the space. It is reasonable to restrict the choice of P_1, P_2 so that (a) they coincide in P, if P is a real point; (b) they are the same for the point $\bar{P}$, conjugate to P, as they are for P, and in one order represent P and in the opposite order $\bar{P}$; (c) their coordinates, referred to a cartesian system, are analytic functions of the coordinates of P and $\bar{P}$, referred to this system; (d) their transforms P_1', P_2' by an arbitrary transformation of a chosen group of real point transformations represent the point P′ into which P is carried by the transformation. Professor Graustein determines, for the group of real collineations and for each of the more important of its subgroups, the affine group, the group of similarity transformations, the group of motions and the group of translations, all the ordered point pairs P_1, P_2, which represent the given point P and have the desired properties. The determination, in the case of each group, depends on the solution of a functional equation.

24. As has been shown by L. L. Dines, Sheffer's five postulates for Boolean algebras in terms of the operation "rejection," while independent in the sense that no one of them is implied by the other four, are not completely independent in the sense defined by Professor E. H. Moore, in as much as the negative of the first postulate implies the third, fourth, and fifth. Dr. Taylor's paper demonstrates the fact that the postulates become completely independent if the first postulate is replaced by one postulating a minimum of four distinct elements instead of two.

25. The present value of a pure endowment, or ${}_nE_x$, seems to be generally regarded as on a par in importance with quite a number of other actuarial symbols of similar nature. Professor Forsyth shows how the symbol may be used to affect ready analyses of numerous situations in actuarial theory which otherwise prove subtly complicated. In particular, he gives concise derivations of formulas used in present day methods of valuation of life insurance policies—such as full preliminary term, modified preliminary term, etc.—wherein the conciseness is due directly to the use of the symbol ${}_nE_x$.

26. The main disadvantage in the use of the mode as an average is the difficulty met in its determination when the statistical data are given in the form of a frequency distribution when the frequencies correspond to class-marks. Professor Forsyth derives a formula for determining the mode in situations like those just mentioned, based upon his formula used for interpolation of ordinates among areas presented in the December (1916) issue of the *Quarterly Publications of the American Statistical Association.*

27. In a previous paper, presented to the Society in December, 1918, Professor Buchanan discussed the orbits which are asymptotic to the straight line equilibrium points in the problem of three finite bodies. The present paper deals with the corresponding orbits for the equilateral triangle equilibrium points.

28. This note shows how to set up certain systems of partial differential equations whose solutions establish all the birational transformations in space of n dimensions. Let $x_1, x_2, \cdots, x_{n+1}$ be homogeneous coordinates. The transformation $\rho x_i' = \varphi_i(x_1, x_2, \cdots, x_{n+1})$ $(i = 1, 2, \cdots, n + 1)$ is birational if the functions φ_i are homogeneous polynomials of the same degree, m, and if the inverse transformation has the same form, its polynomials being of degree m'. Professor Bouton begins by reducing the transformation by means of projections to a normal form in which $\varphi_1 = x_1^m + x_1^{m-2}A_{12} + \cdots + A_{1m}$, $\varphi_i = x_ix_1^{m-1} + x_1^{m-2}A_{i2} + \cdots + A_{im}$ $(i = 2, 3, \cdots, n + 1)$, where A_{ik} is a homogeneous polynomial in $x_2, \cdots, x_{n+1}$, of degree k, and with the inverse in similar form. The differential equations established are those satisfied by

the polynomials A_{ik}. For example, if $m = m' = 2$, the A_{i2} are solutions of the system of partial differential equations

$$2x_iA_{12} + \sum_{s=2}^{n+1} A_{s2}\frac{\partial A_{i2}}{\partial x_s} = 0 \qquad (i = 2, 3, \cdots, n + 1).$$

Conversely, any set of solutions of this system, which are homogeneous polynomials of degree two, will yield a birational transformation, which may, however, be reducible to a projection. There seem to be no references in the literature to this method of setting up all the birational transformations in space of n dimensions.

29. Two real surfaces, in continuous one-to-one point correspondence, with the directed normals in corresponding points parallel, may be said to correspond by a parallel map, or more specifically, by a directly parallel or an inversely parallel map, according as corresponding directions of rotation about corresponding points are the same or opposite. Professor Graustein shows that two real surfaces, which correspond by an inversely parallel, area-preserving map of a particular kind, are translation surfaces with real generators, whereas, if the map is directly parallel and area-preserving of a certain type, the surfaces are translation surfaces with conjugate-imaginary generators. In particular, two real surfaces, applicable by an inversely parallel map, are translation surfaces with real perpendicular generators and hence cylinders; if, on the other hand, the surfaces are applicable by a directly parallel map with the property that the angle between corresponding curves in corresponding points is constant, not 0 or π, they are translation surfaces with conjugate-imaginary minimal curves as generators and are, then, minimal surfaces.

30. Dr. Reynolds proves that if a given linear differential equation of the fourth order has four linearly independent real solutions which satisfy a non-singular quadratic identity, then it may be reduced to a form which may be said to be self-adjoint with respect to the third row of the wronskian of any four linearly independent solutions. The signature of the quadratic identity associated with the equation is then evaluated in terms of the coefficients of the given equation and several theorems concerning the zeros of such an equation are proved.

31. The purpose of Professor Rowe's paper was to make a mathematical statement of the aeronautical radiator problem.

32. Professor Carver and Mrs. King define a property of permutation groups, closely analogous to multiple transitivity, as follows:

A permutation group of degree n is said to be k-ply pseudo-transitive, $1 \leqq k \leqq n - 1$, if there is a set of k letters such that there is at least one permutation of the group sending this set into any arbitrarily chosen set of k letters *in some order*. It is evident that the case $k = 1$ is the same as ordinary simple transitivity; that if a group is k-ply transitive it is k-ply pseudo-transitive, but not conversely; and that if a group of degree n is k-ply pseudo-transitive it is also $(n - k)$-ply pseudo-transitive. Several theorems analogous to those for ordinary multiple transitivity are readily obtained.

Among the groups of degree equal to or less than nine, there are just seven cases of groups having multiple pseudo-transitivity of a multiplicity other than is obviously implied by their ordinary multiple transitivity.

The notion has an incidental application to, and was suggested by, a question concerning the condition for apolarity between two binary forms.

33. In any theory of covariants, we find that closely related to the point of view taken when we try to express all covariants as polynomials in a finite number of covariants is the one taken when we endeavor to express all covariants as rational (though not necessarily integral) functions of a finite number of covariants. Professor Hazlett's paper attacks modular covariants of a single binary form from this second point of view by employing a method which has its origin in Hermite's fundamental memoir on associated forms. The main theorem of the present paper proves that, if f is a binary form of order not congruent to zero modulo p, then any modular covariant of f for the $GF[p^n]$ is expressible (aside from a power of f) as a polynomial in Q and L, where the coefficients of the terms in Q are polynomials in the forms associated with f. From this theorem flow several corollaries, of which one gives a neat method of constructing a modular covariant having a given leader. This theorem, together with the corollaries, is verified for the binary quadratic, modulo 3.

34. Dr. Curtis's paper appears in full in the present number of the BULLETIN.

35. Suppose a curve C of order m and genus p with d double points. Miss Cohen considers the linear series of curves of order αn cutting C n times at each of $\alpha m - p$ fixed points, among which are the d double points, the value of α for m odd being $\frac{1}{2}(m-1)$, and for m even, $\frac{1}{2}(m-2)$. There are exactly n^{2p} such curves cutting C n times at p further points. Suppose such a set of p points to be $A_1, A_2, \cdots, A_p$, the fixed base points to be $F_1, F_2, \cdots, F_{\alpha m-p}$, and any line section represented by L. Then $A_1 + \cdots + A_p + F_1 + \cdots + F_{\alpha m-p} - \alpha L$ is a virtual set that has the properties of an nth of a period for a given set of $\alpha m - p$ base points and the totality of such sets can be shown to represent the totality of $n$ths of periods. For p sufficiently high it is possible to put a curve of order α on $F_1, \cdots, F_{\alpha m-p}$ with p residual intersections $P_1, P_2, \cdots, P_p$. The nth of a period can then be represented by $A_1 + \cdots + A_p - (P_1 + \cdots + P_p)$.

The simplest case is presented by the elliptic parameters on the general cubic.

36. F. Riesz has derived a large part of the Fredholm theory of integral equations for the equation

$$h(x) = f(x) + \lambda T(f),$$

where $T(f)$ is a completely continuous, linear transformation, and he has also proved that every linear transformation, that is to say every linear functional depending on a parameter, can be expressed in terms of a Stieltjes integral such as

$$T(f) = \int_a^b f(y) d_y K(x, y).$$

In the present paper Professor Fischer has proved that if such a transformation is to be completely continuous, it is necessary and sufficient that the variation of $K(x, y)$ in y shall be a bounded function of x, and that whenever $x_1, x_2, \cdots$ are chosen in such a way that $K(x_n, y)$ approaches a unique function of y when n becomes infinite, the variation of $[K(x_n, y) - \text{limit } K(x_n, y)]$ in y shall approach zero. Another necessary and sufficient condition is also found and proved equivalent to this one.

37. The problem of the structure of finite continuous groups consists in finding the structural constants determined by the alternants of operators, generating a group with r essential parameters, of the form

$$X_i = \sum_{k=1}^{n} \xi_k(x) \frac{\partial}{\partial x_k} \qquad (i = 1, 2, \cdots, r).$$

These alternants have the form

$$(X_i, X_j) = \sum_{s=1}^{r} c_{ijs} X_s \qquad (i, j = 1, 2, \cdots, r).$$

The c's which are the structural constants must satisfy two conditions:

$$(1) \qquad c_{ijk} = - c_{jik},$$

$$(2) \qquad \sum_{s=1}^{r}(c_{ijk}c_{kls} + c_{jlk}c_{kis} + c_{lik}c_{kjs}) = 0 \quad (k = 1, 2, \cdots, r).$$

Dr. Zeldin discusses the conditions to be imposed on the group with one exceptional infinitesimal transformation X_r, which would make all c_{irk} equal to zero, where $(i, k = 1, 2, \cdots, r)$.

38. Jensen has recently stated the theorem that if $f(z)$ is a real polynomial and if there are described circles having as diameters the segments joining the pairs of conjugate imaginary roots of $f(z)$, then no non-real root of $f'(z)$ lies outside those circles. Mr. Walsh shows that on any segment of the axis of reals exterior to all of those circles and containing no root of $f(z)$ there is at most one root of $f'(z)$. If any of those circles has on or within it k roots of $f(z)$ and is not interior to nor has a point in common with any other of those circles, it has on or within it not more than $k + 1$ nor less than $k - 1$ roots of $f'(z)$. The paper contains also various other theorems connected with Jensen's theorem.

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