THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE two hundred and tenth regular meeting of the Society was held at Columbia University on Saturday, April 24, 1920, extending through the usual morning and afternoon sessions. The total attendance numbered over one hundred and thirty, and included the following eighty-two members:

Dr. J. W. Alexander, Professor R. C. Archibald, Dr. I. A. Barnett, Professor F. W. Beal, Mr. D. R. Belcher, Professor Suzan R. Benedict, Professor A. A. Bennett, Professor W. J. Berry, Mr. William Betz, Professor Joseph Bowden, Professor R. W. Burgess, Professor B. H. Camp, Professor C. W. Cobb, Professor A. Cohen, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor E. S. Crawley, Dr. W. L. Crum, Professor Louise D. Cummings, Professor L. E. Dickson, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Dr. L. R. Ford, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor Olive C. Hazlett, Professor E. R. Hedrick, Dr. A. A. Himwich, Professor T. R. Hollcroft, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. K. W. Lamson, Mr. Harry Langman, Professor P. H. Linehan, Mr. L. L. Locke, Professor W. R. Longley, Mr. John McDonnell, Professor C. R. MacInnes, Professor H. P. Manning, Dr. A. L. Miller, Professor H. H. Mitchell, Professor R. L. Moore, Professor Frank Morley, Professor G. W. Mullins, Professor W. F. Osgood, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Susan M. Rambo, Professor H. W. Reddick, Professor L. W. Reid, Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor J. E. Rowe, Professor Paul Saurel, Dr. Caroline E. Seely, Professor S. P. Shugert, Professor P. F. Smith, Professor C. M. Sparrow, Dr. J. M. Stetson, Mr. J. J. Tanzola, Professor J. I. Tracey, Mr. H. S. Vandiver, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. A. L. Wechsler, Mr. R. A. Wetzel, Professor H. S. White, Professor J. K. Whittemore, Miss Ella C. Williams, Professor F. B. Williams, Professor A. H. Wilson, Professor Ruth G. Wood, President R. S. Woodward.

President Frank Morley occupied the chair, yielding it to Ex-President R. S. Woodward during the presentation of the papers on relativity at the afternoon session. The Council announced the election of the following persons to membership in the Society: Professor H. S. Everett, Bucknell University; Dr. L. J. Rouse, University of Michigan; Professor Nilos Sakellariou, University of Athens; Mr. H. L. Smith, University of Wisconsin; Professor Eugene Taylor, University of Wisconsin; Professor W. P. Webber, University of Pittsburgh. Thirteen applications for membership in the Society were received.

Professor L. P. Eisenhart was reelected a member of the Editorial Committee of the *Transactions* for a term of three years beginning October 1, 1920. Professor P. F. Smith will retire from the Editorial Committee on October 1, after nine years' service as editor, and Professor G. D. Birkhoff will fill out Professor Smith's unexpired term. Professor Oswald Veblen was appointed representative of the Society in the Division of Physics of the National Research Council for a term of three years beginning July 1, 1920. Professor Veblen's Cambridge Colloquium Lectures on Analysis Situs will be published by the Society late in the fall. Committees were appointed to confer with a committee of the Mathematical Association as to joint plans for future meetings and to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

On the recommendation of the Council it was unanimously voted to incorporate the Society under the membership corporations law of the state of New York. The new form of organization will involve hardly any changes beyond those necessary to comply with legal requirements. The Council will continue as at present constituted. But it is necessary under the law to have an elected Board of Trustees; this will consist of the officers and other members of the Council who are elected by the Society at the annual meetings. Otherwise the Constitution and By-Laws, which have come down almost from the beginnings of the Society and which are a highly efficient instrument of government, well worthy of study, will remain practically as at present.

The Council received a preliminary report from the committee on reorganization, which will make specific recommendations at a later meeting. A report was also received from the committee on the International Mathematical Union, and the formation of an American Section of the Union was approved. The recommendation of the committee on bibliography that a journal of mathematical abstracts be established was approved and the committee was authorized to take steps toward securing the necessary financial support.

The greater part of the afternoon session was devoted to a symposium on the subject of relativity, at which the following papers were presented:

I. "The physical and philosophical significance of the principle of relativity and Einstein's theory of gravitation," by Professor LEIGH PAGE, of Yale University.

II. "Geometric aspects of the Einstein theory," by Professor L. P. EISENHART, Princeton University.

Over fifty members took luncheon together between the sessions, and thirty gathered at the dinner after the meeting.

The regular programme consisted of the following papers:

(1) Professor N. A. COURT: "On a pencil of nodal cubics. Second paper."

(2) Mr. E. L. POST: "Introduction to a general theory of elementary propositions."

(3) Mr. E. L. POST: "Determination of all closed systems of truth tables."

(4) Mr. JESSE DOUGLAS: "The dual of area and of volume."

(5) Professor J. K. WHITTEMORE: "Reciprocity in a problem of relative maxima and minima."

(6) Dr. I. A. BARNETT: "Linear partial differential equations with a continuous infinitude of variables."

(7) Dr. I. A. BARNETT: "Functionals invariant under one-parameter continuous groups in the space of continuous functions."

(8) Professor T. R. HOLLCROFT: "A classification of plane involutions of order four."

(9) Dr. TOBIAS DANTZIG: "A group of line-to-line transformations."

(10) Dr. A. R. SCHWEITZER: "On the iterative properties of the abstract field."

(11) Dr. J. F. RITT: "On the conformal mapping of a region into a part of itself."

(12) Dr. L. R. FORD: "A theorem relative to rational approximations to irrational complex numbers."

(13) Professor L. E. DICKSON: "Report on recent progress in the theory of numbers."

(14) Professor G. D. BIRKHOFF: "Note on the ordinary linear differential equation of the second order."

(15) Professor JOSEPH LIPKA: "The motion of a particle on a surface under any positional forces."

(16) Professor JOSEPH LIPKA: "Note on velocity systems in a general curved space of n dimensions."

(17) Professor J. E. ROWE: "Testing the legitimacy of empirical equations by an analytical method."

(18) Professor OSWALD VEBLEN: "Relations between certain matrices used in analysis situs."

(19) Professor O. D. KELLOGG: "A simple proof of a closure theorem for orthogonal function sets."

(20) Professor C. L. E. MOORE: "Rotation surfaces of constant curvature in a space of four dimensions."

(21) Mr. H. S. VANDIVER: "On Kummer's memoir of 1857 concerning Fermat's last theorem."

(22) Professor NILOS SAKELLARIOU: "A note on the theory of flexion."

Mr. Post was introduced by Professor Keyser; Mr. Douglas by Professor Kasner. The papers of Professor Court, Mr. Post (second paper), Dr. Barnett (first paper), Dr. Dantzig, Dr. Schweitzer, Dr. Ritt, Professor Birkhoff, Professor Lipka, Professor Rowe, Professor Veblen, Professor Kellogg, Professor Moore, Mr. Vandiver, and Professor Sakellariou were read by title.

Abstracts of the papers follow below, being numbered to correspond to the titles in the list above.

1. Continuing the discussion started in his first paper (see this BULLETIN, February, 1920) Professor Altshiller-Court proves that: Two nodal cubics having in common three collinear points, the double point, and the two tangents at this point, are homological, the double point and the base of the three collinear points being the center and the axis of homology, respectively. Several additional properties of the pencil of nodal cubics referred to are derived from this proposition. As an interesting special case of the dual of this proposition is pointed out the following: Two tricuspidal hypocycloids tangent to three concurrent lines are similar and similarly placed, the point of intersection of the three common tangents being the homothetic center of the two curves.

2. In this paper Mr. Post studies in its entirety the deductive system which Whitehead and Russell have developed in Part I, Section A, of their Principia Mathematica. Through the concept of the truth table of a truth function, a uniform method is given for telling whether the assertion of a given propositional function of the system can or cannot be derived from the postulates. By means of this result, a number of properties of the system are obtained, among which is the theorem that any propositional function of the system can either be asserted by means of the postulates or else is inconsistent with them.

Two modes of generalizing the system are considered. One consists in generalizing the primitive functions by means of the truth table concept, and connects up with the work of Sheffer and Nicod. The second or postulational method of generalization is shown to introduce new logical systems.

3. Corresponding to each of the 2^n sets of truth values of the arguments of a truth function $f(p_1, p_2, \cdots, p_n)$, there is a unique truth value of the function. The relation thus set up may be called the truth table of f. Mr. Post considers the systems of truth tables that can be generated by combining arbitrary primitive truth tables and shows that there are 66 different systems generated by primitive tables with no more than three arguments, and 8 infinite families of systems which require tables of four or more arguments. A formula is given for the tables in each system, and it is shown that they include all closed systems of truth tables. These results are applied to the determination of all the ways in which the logical system of truth functions may be generated by independent primitive functions.

4. The object of Mr. Douglas' paper is to develop for the geometry based on the (oriented) line as element a concept analogous to that of area for the geometry based on the point. The main results are as follows:

Denoting by ω, p Hessian line coordinates, the numerical measure of any continuous "mass" of (oriented) lines is equal to $\int\int d\omega dp$. The justification of this definition consists in the fact that the measure so defined possesses the two essential properties of additivity and independence of rigid motion.

The boundary lines of any continuous "mass" envelop, in the usual case, an ordinary closed curve. The double integral $\int\int d\omega dp$ is equal to $\int pd\omega$ taken over this boundary, which last is equal to the length of the bounding curve. The two geometric magnitudes customarily associated with any closed curve, namely the area and the perimeter, thus appear as duals of one another.

By means of a representation of the lines of the plane upon the points of a cylinder as follows: the (oriented) line of Hessian coordinates ω, p corresponds to the point of longitude ω and altitude p, the notion of measure is then extended to non-continuous line sets.

In the latter part of the paper the analogous ideas for three-dimensional space are considered. The measure of a "mass" of planes is found to be $\int\int\int \cos\varphi\, d\theta\, d\varphi\, dp$, where θ, φ, p are polar plane coordinates. On interpreting this integral geometrically, the plane measure of a polyhedron turns out to be equal to $\frac{1}{2}\Sigma e_i A_i$, where e_i is the length of any edge and A_i the dihedral angle of the (oriented) planes meeting in that edge.

5. Professor Whittemore's paper considers the following questions: If $\varphi(x, y) = u$ and $\psi(x, y) = v$ are single-valued real functions of two real variables x, y, having continuous partial derivatives of the first and second orders in the neighborhood of $P(x_0, y_0)$, A: When is $\varphi(x_0, y_0) = u_0$ a maximum or a minimum of u with the condition $\psi = v_0$? B: When is v_0 a maximum or a minimum of v with the condition $\varphi = u_0$? If there are extremes in both A and B, when are they like extremes?

It is proved for the general case, where not both first derivatives of either φ or ψ vanish at P, that a necessary condition for an extreme for both A and B is that the two curves $\varphi = u_0$ and $\psi = v_0$ be tangent at P; that a sufficient condition for an extreme in both problems is that these curves do not osculate at P; when both these conditions are satisfied that the extremes in A and B are like or unlike as φ_y and ψ_y have unlike or like signs at P.

The exceptional case, where both first derivatives of φ or of ψ or of both functions vanish at P, is considered; necessary and sufficient conditions for a maximum or a minimum are obtained in so far as this is possible by use of the first and second derivatives. In some cases the discussion fails to establish such conditions. For several cases it is shown that, in A, u_0 is a maximum or minimum as $\psi_{yy}H + 2\varphi_{yy}\Delta\psi$ is negative or positive at P, where

$$H = \varphi_{xx}\psi_{yy} + \psi_{xx}\varphi_{yy} - 2\varphi_{xy}\psi_{xy}, \qquad \Delta\psi = \psi_{xy}^2 - \psi_{xx}\psi_{yy}.$$

In several of the cases where u_0 and v_0 are extremes in A and B respectively it is proved that they are like or unlike extremes as H is positive or negative.

The invariant character of the conditions established for any change of variables, not singular at P, is considered. Simple examples illustrating the theory are given.

6. In a previous paper Dr. Barnett has given an existence theorem for solutions of linear non-homogeneous partial differential equations with a continuous infinitude of unknowns. In the present paper the theory there developed is applied to linear homogeneous equations. It is found convenient to use the theory of implicit functional equations as developed by Lamson in his Chicago dissertation.

7. Dr. Barnett considers one-parameter continuous groups of transformations $\bar{u}(x) = F[x, u(\xi), t]$ where F is a functional operation yielding for each continuous function of a given range and for each value of the parameter t, another continuous function. Infinitesimal transformations are defined and invariants of the groups are considered. These invariant functionals $G[u(x)]$ are found to be solutions of certain functional equations given in the preceding paper. Examples which are analogues of some of the well-known groups in n-space are studied.

8. Professor Hollcroft discusses the conditions imposed upon two planes when an algebraic (1,4) point correspondence is established between them. The particular correspondence given is defined by a curve of order n with a point P of multiplicity $n - 4$ and a line pencil on P, and is one of ten independent types of plane involutions of order four. An independent type is one that cannot be reduced to another by any birational transformation.

9. In a previous communication to the Society (April, 1916,

Chicago Section) Dr. Dantzig considered certain properties of point-to-point transformations and particularly a certain system of ∞^2 conics which he called the indicating system of the transformation. These properties are here extended to line-to-line transformations, and a special study is made of those transformations which admit a focal line, i. e., a bearer of an invariant range of points. Especially interesting is the case where the line at infinity is a focal line. The indicating system then consists of parabolas. These transformations form a group, the parabolic group G. The fundamental property of a transformation belonging to G is the following: Let l and $\bar{l}$ be two corresponding lines meeting in a point λ; let moreover τ and $\bar{\tau}$ be the points where the indicatrix I touches l and $\bar{l}$ respectively. If c and c' are any two curves touching l in p and p', while their transforms $\bar{c}$ and $\bar{c}'$ touch $\bar{l}$ in $\bar{p}$ and $\bar{p}'$, then the ratio

$$\kappa = \frac{\overline{pp'}}{pp'} = \frac{\bar{\tau}\lambda}{\tau\lambda}$$

and consequently is invariant along the line l. The transformations which leave the ratio κ invariant throughout the plane form a subgroup S of G. However, the subclass of S admitting the same value of κ, say κ_0, does not form a subgroup of S, unless $\kappa_0 = 1$. In this latter case we obtain the Scheffers equilong group.

The rest of the paper is taken up by the generalization of these properties to the case where the focal line is of general position and by the interpretation of the corresponding Scheffer's subgroup. There is also given a general determination of the parabolic transformation admitting an axis and a detailed study of the so-called Laguerre group of line-to-line transformations.

10. In a previous paper Dr. Schweitzer constructed a set of postulates (with explicit reference to iterative compositions) for an abstract field in terms of two undefined quasi-transitive functions, $f(x, y)$ $[x - y]$ and $f_1(x, y)$ $[x/y]$ corresponding to postulates for an abstract group in terms of a relation concretely interpreted by $x \cdot y^{-1}$. In another paper, primarily in reference to the abstract field, he considered quasi-transitive functional equations of the type

$$f\{\lambda f(x, y), \mu f(x, z)\} = \nu f(z, y). \tag{1}$$

A solution of (1) for $\lambda(x) = \mu(x)$, $\nu(x) = x$ is

$$f(x, y) = X^{-1}\{c'X(x)/X(y) + c''\}, \quad \lambda(x) = X^{-1}\{X(x) - c''\},$$

where c' and c'' are constants. Second, a solution of (1) for $\lambda(x) = \mu(x) = x$ is $f(x, y) = X^{-1}\{cX(x) - cX(y)\}$, $\nu(x) = X^{-1}\{cX(x)\}$. If the "arbitrary" constant c is interpreted as a parameter, say t, then special cases of the latter function are

$$f_1(x, y, t_1) = (x - y)\cdot t_1, \quad f_2(x, y, t_2) = (x - y)/t_2,$$

each of which may serve abstractly as a ternary generating relation for an abstract field.*

11. Let a domain composed of a continuum plus its frontier be mapped conformally, in a one-to-one manner, into a part of itself, so that every frontier point of the original domain becomes an interior point. Dr. Ritt shows that, under the transformation, one point and only one point stays fixed. The derivative of the mapping function is less than unity in modulus at the fixed point. If the transformation is iterated, the consequents of every point of the original domain approach the fixed point. For the case of a simply connected domain this theorem has already been given by Julia, without assuming that the transformation is one-to-one. Julia's method, which is entirely different from that used in the present paper, does not admit of extension to a general continuum.

12. Let ω be any irrational complex number, and consider the inequality

$$\left|\omega - \frac{p}{q}\right| < \frac{k}{|q|^2},$$

where k is real and p/q is a rational fraction (i. e., p and q are each of the form $m + ni$, where m and n are real integers). Dr. Ford proves the following theorem: If $k = 1/\sqrt{3}$ there are infinitely many fractions satisfying the given inequality; if $k < 1/\sqrt{3}$ there exists an everywhere dense set of ω's for each of which the inequality holds for only a finite number of fractions.

* Cf. this BULLETIN, June, 1918, p. 428; March, 1915, p. 295; October, 1915, p. 4; October, 1914, p. 28; Nov., 1918, pp. 58, 59.

13. Professor Dickson made a report on the literature on the theory of numbers during the past decade, prepared in accordance with a decision by the members of the Division of Physical Sciences of the National Research Council to prepare reports of recent progress in various branches of astronomy, mathematics, and physics. While most of these reports will probably cover only the past year, it seemed best in the case of a mathematical subject to cover the past decade, especially in view of the present difficulty of obtaining complete references in a field of mathematics. The present task was simplified by the fact that the writer had already taken great pains to cover the literature to date in his History of the Theory of Numbers, now in course of publication. During the past decade there have appeared fully 1,600 papers, notes and books on this subject. Those relating to divisibility and primality have been considered in volume I of the History, and are only cited en masse in this report in connection with the still more recent papers. In the case of Diophantine analysis (the subject of volume II, in press), only the more important recent papers are cited in this report. But there is given a rather complete list of recent papers on the remaining topics, since their appearance in volume III will be delayed at least a year.

14. Professor Birkhoff shows that it is possible to develop the theory of the ordinary linear differential equation of the second order with real independent variable in such wise that the conditions imposed are both necessary and sufficient throughout. In the usual formulation of the theory the conditions used are not of this type.

15. The complete system of trajectories of a particle constrained to move on a surface under any positional forces form a family of ∞^3 curves, one through each point in each direction for each value of the speed. Professor Lipka derives some of the geometric properties of such a system of curves. One of the simplest of these properties may be stated as follows: in each direction through a given point there passes one trajectory which hyperosculates its circle of constant geodesic curvature; the locus of the centers of geodesic curvature of the ∞^1 hyperosculating trajectories obtained by varying the direction through the point is a conic passing through the

given point in the direction of the force acting at that point. If the force is conservative, the complete systems of brachistochrones and catenaries on a surface also form families of ∞^3 curves. Some properties of these systems are derived, and a comparative study of the three systems, trajectories, brachistochrones and catenaries, is made.

16. In a previous paper,* Professor Lipka characterized certain systems of curves, termed velocity systems, in a general curved space of n dimensions, by means of their geometric properties. In the present note, the author points out a dynamical problem which leads to the same system of curves.

17. After the form of an empirical equation is assumed, the one equation of this assumed type which satisfies the data with least error may be obtained. But the question which remains unanswered is this: Have the data been forced to satisfy an equation of this particular type by virtue of the assumed form of the equation, or do they satisfy an equation of this type naturally? In Professor Rowe's paper an analytical method is developed which enables us to ascertain what type of equation the data (which are known to be correct) most naturally satisfy. The theory involved is a very great addition to, and is not in any sense a substitution for, any good judgment that might be used originally in assuming the most probable type of equation.

18. In Professor Veblen's paper, the matrix giving the number of intersections, counted with regard to sign, of a complete set of i-dimensional manifolds in an n-dimensional manifold, M_n, with a complete set of $(n - i)$-dimensional manifolds of M_n is called an intersection matrix. It is shown how to obtain these matrices from the matrices giving the relations between the oriented j-cells and the oriented $(j - 1)$-cells on their boundaries by means of algebraic operations.

19. Let $n(x)$ be the set of normal functions arising from a differential equation of the Sturm type with general self-adjoint homogeneous linear boundary conditions. Professor

* "Natural families of curves in a general curved space of n dimensions," *Trans. Amer. Math. Society*, vol. 13, pp. 77–95.

Kellogg's note gives a simple proof that if this set is closed with respect to continuous functions, it is also closed with respect to summable functions which are not null functions. Hilbert and others have shown that they are closed with respect to continuous functions, and they are therefore closed with respect to the broader class. The interest of the note lies rather in the method of proof than in the results, which are largely already known.

20. In this note Professor Moore discusses the forms of curves that will generate surfaces of constant curvature when rotated by the special rotations leaving a doubly infinite number of planes invariant.

21. In an article in the *Mathematische Abhandlungen* of the Berlin Academy for the year 1857, pages 41–74, Kummer essayed to prove that the relation

$$x^p + y^p + z^p = 0 \tag{1}$$

could not be satisfied in integers, when p is an odd prime not satisfying three given conditions. Based on this result, the conclusion that (1) is impossible for all p's less than 100 was derived by him. In the present paper Mr. Vandiver points out that Kummer made several errors in his argument, which vitiate his results. The paper will appear in the *Proceedings of the National Academy of Sciences*.

F. N. COLE,
Secretary.

STIELTJES DERIVATIVES.

BY PROFESSOR P. J. DANIELL.

THE fundamental theorem for the derivative with respect to a function of limited variation is difficult to prove in the case of several dimensions, and no attempt is made here to consider the most general derived numbers. In place of the method used by the author for one dimension we shall use methods and ideas due to C. de la Vallée Poussin and W. H. Young.*

* C. de la Vallée Poussin, Intégrales de Lebesgue. Paris, 1916, pp. 61–73. W. H. Young, *Proc. London Math. Society* (1914), vol. 13, p. 109. P. J. Daniell, *Transactions Amer. Math. Society* (1918), vol. 19, p. 353.