NOTE ON A GENERALIZATION OF A THEOREM OF BAIRE.

BY PROFESSOR E. W. CHITTENDEN.

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IT is the purpose of this note to call attention to the following generalization of a celebrated theorem of Baire:

THEOREM: *Let P be a perfect set in a complete, separable, metric space S.* Then a necessary and sufficient condition that a function defined on P be of class* 1 *in the Baire classification of functions is that the function be at most point-wise discontinuous on every perfect subset of P.*

The necessity of the condition has been established by Hausdorff.†

A satisfactory proof of this theorem can be obtained from a proof given by Vallée-Poussin‡ for the corresponding theorem in space of n dimensions by making appropriate changes in terminology, since Vallée-Poussin makes no essential use of the special properties of space of n dimensions in his argument. The methods to be followed in generalizing this proof of Vallée-Poussin are sufficiently indicated in the treatise of Hausdorff already cited and in the memoirs of Fréchet.§

There is, however, one point which requires special consideration. Vallée-Poussin introduces an auxiliary function with the following definition: If M is a point of S; $H_1, H_2, \cdots, H_n$ are closed sets such that no two have a common element; $\delta_1, \delta_2, \cdots, \delta_n$ are the respective distances of $H_1, H_2, \cdots, H_n$ from M; and $a_1, a_2, \cdots, a_n$ are constants; then

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* See Hausdorff, Grundzüge der Mengenlehre, Veit und Co., Leipzig, 1914, pp. 211, 315. The word "vollständig" is here translated "complete." A complete set admits a generalization of the theorem of Cauchy in the sense of Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matem. di Palermo*, vol. 22 (1906), pp. 1–74.

† Hausdorff, loc. cit., p. 388.

‡ Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire, Gauthier-Villars, Paris, 1916, pp. 105–125.

§ As an example of the application of the theory of transfinite ordinal numbers in the present situation, see Fréchet, "Les ensembles abstraits et le calcul fonctionnel," *Rendiconti del Circolo Matem. di Palermo*, vol. 30 (1910), pp. 5–10.

$$\varphi(M) = \frac{a_1/\delta_1 + a_2/\delta_2 + \cdots + a_n/\delta_n}{1/\delta_1 + 1/\delta_2 + \cdots + 1/\delta_n},$$

except when M is in H_i; then $\varphi(M) = a_i$ $(i = 1, 2, 3, \cdots, n)$.*

The function $\varphi(M)$ is continuous in space of n dimensions. I wish to show that it is continuous in the space S. It is sufficient to show that the distance δ of a point M from a closed set H is a continuous function of M, vanishing on H.

Let (A, B) denote the distance between two points A, B of the metric space S. Then we have the following fundamental property of distance: If A, B, C are any three points of S,

$$(A, B) \leqq (A, C) + (C, B).\dagger$$

If H is a closed set the distance (M, X), where X is in H, has a minimum δ at a point X_0. We call $\delta = (M, X_0)$ the distance from M to H. Let M' be any other point of H, and $\delta' = (M', X_0')$ be the corresponding distance from H. Then we have

$$\begin{aligned}\delta = (M, X_0) \leqq (M, X_0') &\leqq (M, M') + (M', X_0') \\ &\leqq (M, M') + \delta'.\end{aligned}$$

Similarly, $\delta' \leqq \delta + (M, M')$. Therefore $|\delta - \delta'| \leqq 2(M, M')$, and the continuity of δ is established.

UNIVERSITY OF IOWA,
IOWA CITY, IOWA.
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CERTAIN ITERATIVE CHARACTERISTICS OF BILINEAR OPERATIONS.

BY DR. NORBERT WIENER.

Introduction.

IN a recent paper‡ the author has developed the necessary and sufficient condition that a bilinear operation in two variables should generate by iteration all rational operations with

* Vallée-Poussin, loc. cit., p. 118.

† Hausdorff, loc. cit., p. 211. See also Fréchet, "Les notions de limite et de distance," *Transactions Amer. Math. Society*, vol. 19 (1918), p. 54.

‡ *Annals of Mathematics*, vol. 21, No. 3 (March, 1920), pp. 157–165.