THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and twelfth regular meeting of the Society was held at Columbia University on Saturday, October 30, 1920, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members of the Society:

Dr. J. W. Alexander, Professor A. A. Bennett, Professor F. N. Cole, Dr. Tobias Dantzig, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Mr. Philip Franklin, Dr. T. H. Gronwall, Dr. C. M. Hebbert, Dr. A. A. Himwich, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. K. W. Lamson, Mr. Harry Langman, Dr. H. F. MacNeish, Professor Frank Morley, Professor W. F. Osgood, Dr. G. A. Pfeiffer, Dr. E. L. Post, Professor H. W. Reddick, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor H. W. Tyler, Professor Oswald Veblen, Professor J. H. M. Wedderburn, Mr. R. A. Wetzel, Professor H. S. White, Professor J. K. Whittemore, Dr. T. S. Yang.

President Morley occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. P. M. Batchelder, University of Texas; Miss Vevia Blair, Horace Mann School; Mr. E. H. Carus, La Salle, Ill.; Mr. W. E. Cederberg, University of Wisconsin; Mr. R. P. Conkling, Newark Technical School; Mr. P. H. Evans, Northwestern Mutual Life Insurance Company, Milwaukee, Wis.; Mr. B. L. Falconer, U. S. Civil Service Commission, Boston, Mass.; Mr. J. A. Foberg, Crane Junior College, Chicago, Ill.; Dr. Gladys E. C. Gibbens, University of Minnesota; Professor L. E. Gurney, University of the Philippines; Professor Archibald Henderson, University of North Carolina; Miss Jewell C. Hughes, University of Arkansas; Miss Claribel Kendall, University of Colorado; Mrs. M. I. Logsdon, University of Chicago; Mr. R. L. McNeal, General Motors Laboratories, Detroit, Mich.; Mr. H. L. Olson, University of Michigan; Professor Leigh Page, Yale University; Capt. H. W. Rehm, Aberdeen Proving Ground, Md.; Mr. Irwin Roman, Northwestern University; Mr.

Raleigh Schorling, Lincoln School, New York City; Mr. E. L. Thompson, Junior College, Joliet, Ill.; Dr. Bird M. Turner, University of Illinois. Four applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. The Treasurer of the Society to be elected at the annual meeting was made curator of all property belonging to the Society.

It was announced that the next summer meeting of the Society will be held at Wellesley College. The Mathematical Association of America, which will hold its winter meeting at Chicago, will meet with the Society at Wellesley.

The following papers were read at the October meeting:

(1) Mr. H. S. VANDIVER: "On Kummer's memoir of 1857 concerning Fermat's last theorem. Second paper."

(2) Professor R. L. BORGER: "On total differentiability."

(3) Professor ELIZABETH LESTOURGEON: "Minima of functions of lines."

(4) Professor JOSEPH LIPKA: "Complete geometric characterization of the dynamical trajectories on a surface for any positional field of force."

(5) Professor JOSEPH LIPKA: "Complete geometric characterization of brachistochrones, catenaries, and velocity curves on a surface."

(6) Professor DUNHAM JACKSON: "On the convergence of certain polynomial approximations."

(7) Dr. J. F. RITT: "On algebraic functions which can be expressed in terms of radicals."

(8) Professor A. A. BENNETT: "The Schwarz inequality for a given symmetrical convex region and given bilinear form."

(9) Professor EDWARD KASNER: "Determination of an Einstein gravitational field by means of the paths of free particles."

(10) Professor O. E. GLENN: "An algorism for differential invariant theory."

(11) Dr. T. H. GRONWALL: "Some inequalities in the theory of functions of a complex variable."

(12) Dr. W. L. G. WILLIAMS: "Fundamental systems of formal modular seminvariants of the binary cubic."

Dr. Williams' paper was communicated to the Society through Professor L. E. Dickson. In the absence of the authors the papers of Mr. Vandiver, Professor Borger, Professor LeStourgeon, Professor Lipka, Professor Jackson, Professor Glenn and Dr. Williams were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The first paper under the present title appeared in the *Proceedings of the National Academy* for May, 1920. In it, Mr. Vandiver mentions four theorems of Kummer's, which he numbers I to IV. It is shown that Kummer's arguments for each theorem are either deficient or inaccurate. In the present paper, the work of Kummer is modified in such a way as to yield proofs of theorems I and IV.

2. Professor Borger's paper contains a necessary and sufficient condition that a function of two independent variables may possess a total differential in any point where its partial derivatives exist and are finite. The extension to n variables is also indicated.

For a function of two variables this condition may be stated as follows: If the limits

$$\lim_{(k,\, h \doteq 0)} \frac{f(x+h,\, y+k) - f(x,\, y+k)}{h},$$

$$\lim_{(h,\, k \doteq 0)} \frac{f(x+h,\, y+k) - f(x+h,\, y)}{k}$$

exist; and if

$$\lim_{(k,\, h \doteq 0)} \frac{f(x+h,\, y+k) - f(x,\, y+k)}{h} = f_x,$$

$$\lim_{(h,\, k \doteq 0)} \frac{f(x+h,\, y+k) - f(x+h,\, y)}{k} = f_y,$$

the function $f(x, y)$ is totally differentiable in the point (x, y) and conversely.

The notation $\lim_{(h,\, k \doteq 0)}$ signifies that h and k approach zero so that the absolute value of the first variable in the parenthesis is less than or equal to that of the second.

3. Fréchet has defined first and second differentials for a function of a line or functional $F(\lambda)$. If $\lambda_0(x) + \eta(x)$ is a variation of $\lambda_0(x)$, the first differential is a linear functional $L(\eta)$ and the second is expressible in the form $B(\eta, \eta)$, where $B(u, v)$ is a bilinear functional in the independent arguments $u(x)$ and $v(x)$. The definitions of Fréchet apply only to functionals and differentials having continuity of order zero. Professor Le Stourgeon shows that if the continuity is of the first order, as in the case of the integrals of the calculus of variations, the differentials $L(\eta)$ and $B(\eta, \eta)$ are expressible in the forms

$$L(\eta) = \int_a^b \eta(x)du(x) + \int_a^b \eta'(x)du_1(x),$$

$$B(\eta, \eta) = \int_a^b\int_a^b \eta(x)\eta(y)d_{xy}p(x, y)+2\int_a^b\int_a^b \eta(x)\eta'(y)d_{xy}q(x,y) + \int_a^b\int_a^b \eta'(x)\eta'(y)d_{xy}r(x, y).$$

If the functional $F(\lambda)$ has a minimum at λ_0, then it is proved that u and u_1 must satisfy an equation of the form

$$u_1(x) - \int_a^x u(x)dx = kx + l,$$

where k and l are constants. Furthermore, under certain restrictions, a necessary condition for a minimum analogous to the Jacobi condition of the calculus of variations is deduced. It is proved that when $F(\lambda_0)$ is a minimum the equation

$$\int_a^b [u(y)d_yq(y, x) + u'(y)d_yr(x, y)] - \int_a^x\int_a^b [u(y)d_yp(x, y) + u'(y)d_yq(x, y)]dx = kx + l$$

can have no solution $u(x)$, except $u(x) \equiv 0$, vanishing at $x = a$ and a point $x = x'$ between a and b.

The paper appeared in the October issue of the *Transactions*.

4. In this paper, Professor Lipka derives five geometric properties of the system of trajectories generated by the motion of a particle on any constraining surface under any

positional forces. The properties involve the system of trajectories, certain plane systems associated with them by projection into the tangent planes, the lines of force, and isothermal nets of curves, and are stated in terms of geodesic curvature, osculating and hyperosculating geodesic circles, and osculating parabolas. It is shown that the five geometric properties completely characterize the system of trajectories in the sense that any triply infinite system of curves on a surface which possesses these five properties may be considered as generated by the motion of a particle in a unique field of force.

5. This paper presents a study (analogous to that made for dynamical trajectories under positional forces) of certain systems of curves on a surface termed "n" systems—which include brachistochrones, catenaries, velocity curves, and dynamical trajectories as special cases. The field of force is conservative. Professor Lipka derives five geometric properties and shows that these completely characterize an "n" system, in the sense that any triply infinite system of curves on a surface which possesses these five properties may be considered as an "n" system—system of dynamical trajectories, brachistochrones, catenaries, velocity curves, etc., depending on the value of "n."

6. In this paper, Professor Jackson extends to a problem of polynomial approximation the method of treatment used in a paper presented to the Society at the summer meeting in Chicago, on the corresponding trigonometric case. It is assumed that $f(x)$ is a given function, continuous for $a \leqq x \leqq b$, and that $P_{mn}(x)$ is the polynomial of the nth degree which gives the best approximation to $f(x)$ over the interval, in the sense of the integral of the mth power of the absolute value of the error. The conclusions reached are not exactly parallel to those in the trigonometric case. It is found that, when m is held fast and n is allowed to become infinite, $P_{mn}(x)$ converges uniformly to the value of $f(x)$ for $a \leqq x \leqq b$, provided that $\lim_{\delta=0} \omega(\delta)/\sqrt[m]{\delta^2} = 0$, where $\omega(\delta)$ is the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. The condition as stated is significant only for $m > 2$; its form is appropriately modified for $1 \leqq m \leqq 2$. Somewhat less stringent conditions are found for convergence in the interior of the interval.

7. Let $F(w, z) = 0$ be an irreducible algebraic equation of prime degree n in w. Let the number of values of z for which the Riemann surface for w has branch points be q. Dr. Ritt shows that if w can be expressed in terms of radicals, then

$$q \leqq 4 + \frac{4p}{n-1},$$

where p is the genus of the given algebraic relation.

It can be shown from this inequality that for every genus greater than zero, there is an upper bound to those prime numbers n which can serve as the degree of w in an algebraic relation which permits w to be expressed in terms of radicals.

For the special and interesting case of $p = 0$, we have $q = 2$, 3 or 4. If the monodromy group contains a cyclic substitution, we must have $q = 2$ or $q = 3$. In that case the Riemann surface for w can be changed by a linear transformation either into the surface for $z^{1/n}$ or into the surface for $\cos((\cos^{-1} z)/n)$. It follows that the only polynomials of prime degree whose inverses can be expressed in terms of radicals are $(az + b)^n + c$ and $cf_n(az + b) + d$, where $\cos nz = f_n(\cos z)$.

For $q = 4$, the equations are those which appear in the transformations of prime order of the elliptic functions.

If $q = 3$ and the monodromy group contains no cyclic substitution, we must have $n = 3r + 1$, $4r + 1$ or $6r + 1$, to each of which cases corresponds a special class of equations.

8. Starting with an arbitrary symmetrical convex region, in a domain of a linear set of elements, Professor Bennett constructs a norm, having a linearly homogeneous property, and satisfying a "triangle inequality." For a dual set of variables, an inner product and a dual norm are obtained. These will in every case satisfy the Schwarz inequality familiar in integral equations. The norms thus obtained are determined completely by the character of the given symmetrical convex region.

9. Professor Kasner shows that a four-dimensional manifold obeying Einstein's gravitational equations is essentially determined by its geodesics (paths of particles). Two Einstein manifolds cannot admit geodesic representation without being equivalent (applicable). Hence a complete knowledge of the

orbits of planets in the solar gravitational field determines that field completely, and therefore makes it possible to predict the shape of the light rays. The converse determination, by light rays alone, was discussed in one of the "Five notes on relativity" read at the summer meeting in Chicago. Thus we have in particular a two-fold connection between the motion of the perihelion of Mercury (43″), and the deflection of a light ray (1.7″).

10. The initial point of view in Professor Glenn's paper is obtained from the poles of the transformation on the differentials which one derives from relations $x_r = x_r(y_1, y_2)$ $(r = 1, 2)$ by which a differential quantic f is transformed. These are zeros of linear differential covariants $df_{\pm 1}$ appertaining to a domain R defined by certain irrational expressions in the functions x_r and their derivatives. The coefficients of the expansion of f in terms of $df_{\pm 1}$ as arguments are relative differential parameters, here called invariant elements, appertaining to R. Moreover every differential parameter (or covariant) is, in R, a function of the invariant elements and their derivatives, which accordingly afford a basis of classification for the known types of parameters and for several new types belonging to various domains. The methods of the paper give a direct approach to finiteness theorems and proofs in the subject. A complete system of 31 parameters of a so-called orthogonal type for the differential quantic of order six is produced in the paper, and, in addition, certain more general systems of an extended orthogonal type.

11. In this paper, Dr. Gronwall establishes inequalities analogous to that of Carathéodory for the cases where the real or imaginary part, or both, of a power series convergent in the unit circle is bounded above or below, or both ways.

12. Dr. Williams proves certain general theorems regarding formal modular seminvariants of the binary cubic, modulo p, a prime. A method of deriving a fundamental system of formal seminvariants for any particular prime is outlined and the method is applied to the cases $p = 5$ and $p = 7$. The results for $p = 5$ agree with those found by L. E. Dickson and published in his Madison Colloquium Lectures.

F. N. Cole,
Secretary.