where the last reduction was made by interchanging integrations in (23) and by comparing the result with (22). The interchange was permissible* because of our present hypotheses. Since $[z_i(x') - z_i(x)]$ is the ith Fourier coefficient of the left side of (20), we have completed the proof of the theorem.

In a later paper the author will consider applications of the present results in the theory of functionals whose arguments are summable functions.

UNIVERSITY OF MINNESOTA,
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NOTE ON MINIMAL VARIETIES IN HYPERSPACE.

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1. IT is known that a necessary and sufficient condition that a surface of two dimensions in hyperspace be minimal is the vanishing of the vector mean curvature.† It is the purpose of this note to show that mean curvature of a variety V_m in a space of n dimensions can be defined in the same way and that its vanishing is a necessary and sufficient condition that V_m be minimal. I shall use the absolute calculus, since one of its chief merits is the ease with which invariants can be written down. In fact the very form of an expression shows whether or not it is invariant. Enough vector analysis is used to simplify the form of the expressions.

The variety V_m can be written vectorially in the form

$$y = y(x_1, x_2, \cdots, x_m).$$

Then

$$ds^2 = dy \cdot dy = \sum_1^m a_{rs} dx_r dx_s.$$

If we write

$$y_r = \frac{\partial y}{\partial x_r},$$

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* Cf. de la Vallée Poussin, loc. cit., p. 53.

†Wilson and Moore, "Differential geometry of two-dimensional surfaces in hyperspace," *Proceedings of the Amer. Acad.*, vol. 52 (1916).

we see at once that

(1) $$a_{rs} = y_r \cdot y_s.$$

Hence we may write

$$dy \cdot dy = \Sigma y_r \cdot y_s dx_r dx_s.$$

Differentiating (1) covariantly with respect to the first fundamental form,

(2) $$\varphi = \Sigma a_{rs} dx_r dx_s,$$

remembering that the first covariant derivative of the coefficients of the fundamental form is zero, we obtain

$$y_{rt} \cdot y_s + y_r \cdot y_{st} = 0,$$

and since this relation holds for any r, s, t, we may write

$$y_{ts} \cdot y_r + y_t \cdot y_{sr} = 0,$$

$$y_{rs} \cdot y_t + y_r \cdot y_{st} = 0.$$

The vector y is a function of x_1, x_2, $\cdots$, x_m. Since the second covariant derivative of a function is symmetric, we have

$$y_{rs} = y_{sr}.$$

From the last four equations we have

$$y_r \cdot y_{st} = 0, \quad r, s, t = 1, 2, \cdots m.$$

These equations show that y_{st} is perpendicular to y_r, and it is therefore perpendicular also to the tangent m-space of V_m.

Since the y_{rs} are vectors lying in the normal space, we can express them as linear functions of mutually perpendicular vectors in that space, which is of $n - m$ dimensions. Thus we find

(3) $$y_{rs} = b_{1/rs} z_1 + b_{2/rs} z_2 + \cdots + b_{(n-m)/rs} z_{n-m}.$$

We have also the following relations:

(4) $$z_i \cdot z_j = 0, (i \neq j); \quad z_i \cdot z_i = 1, \quad z_i \cdot y_s = 0.$$

Differentiating these covariantly, we have

(5) $$z_{i/s} \cdot z_j + z_i \cdot z_{j/s} = 0; \quad z_{i/r} \cdot y_s + z_i \cdot y_{rs} = 0.$$

Then, multiplying (3) through by z_i, we have

$$z_i \cdot y_{rs} = b_{i/rs} = - z_{i/r} \cdot y_s.$$

2. *The Second Fundamental Forms.*—The coefficients $b_{i/rs}$ are generalizations of the coefficients of the second fundamental form of a surface in n-space.* They may be taken as the coefficients of $n - m$ fundamental forms which we can combine into a vector fundamental form

$$(6) \qquad \psi = \Sigma y_{rs} dx_r dx_s.$$

We now consider the m systems of curves

$$\lambda_i^{(r)} = \frac{dx_r}{ds_i}, \qquad (i = 1, 2, \cdots m)$$

where ds_i is the element of arc along the curve λ_i. Multiplying by the proper factor, we can make the quantities λ_i satisfy the relation

$$\Sigma_r \lambda_i^{(r)} \lambda_{i/r} = 1.$$

We shall now assume that this relation is satisfied. Such systems are called unit systems. The λ's are called the coordinates of the congruence. If the congruences are orthogonal, the coordinates satisfy the relations

$$\Sigma \lambda_{i/r} \lambda_j^{(r)} = \epsilon_{ij},$$

$$\Sigma \lambda_{i/r} \lambda_i^{(s)} = \epsilon_{rs},$$

where $\epsilon_{kl} = 0$, if $k \neq l$, $\epsilon_{kk} = 1$. In terms of the coordinates, we can write the coefficients of the first fundamental form as follows:†

$$a_{rs} = \Sigma_i \lambda_{i/r} \lambda_{i/s}.$$

The coefficients of the second fundamental form can be written as follows:

$$y_{rs} = \Sigma_{i,\,j} \, \omega_{ij} \lambda_{i/r} \lambda_{j/s},$$

where ω_{ij} is a vector. The invariant vectors

$$(7) \qquad \xi_i = \Sigma_r \lambda_i^{(r)} y_r = \Sigma_r \lambda_{i/r} y^{(r)}$$

are mutually orthogonal unit vectors tangent to the curves whose covariant coordinates are $\lambda_{i/r}$. Solving these equations for y_r and $y^{(r)}$ we find

$$y_r = \Sigma_i \xi_i \lambda_{i/r}, \quad y^{(r)} = \Sigma \xi_i \lambda_i^{(r)}.$$

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* See Wilson and Moore, loc. cit., p. 308.
† Wilson and Moore, loc. cit., §13, p. 289, and §§26, 27, pp. 310–311.

We now introduce Ricci's coefficients of rotation*

$$\gamma_{hkl} = \Sigma_{r,s}\lambda_{h/rs}\lambda_k^{(r)}\lambda_l^{(s)}.$$

Solving these equations for $\lambda_{h/rs}$, we find

(8) $$\lambda_{h/rs} = \Sigma_{i,j}\gamma_{hij}\lambda_{i/r}\lambda_{j/s}.$$

Substitution of this value in the covariant derivative of (7) gives

(9) $$\xi_{i/r} = \Sigma_j\omega_{ij}\lambda_{j/r} + \Sigma_{j,l}\gamma_{ijl}\lambda_{l/r}\xi_j.$$

The condition that the congruence λ_i is a congruence of geodesics† is

$$\gamma_{jii} = 0.$$

3. *The Invariants* ω_{ii}. The curvature of a curve of the congruence λ_i which passes through a given point is obtained by differentiating ξ_i with respect to the arc length s_i. Thus we find

$$C_i = \frac{d\xi_i}{ds_i} = \Sigma_r\frac{\partial\xi_i}{\partial x_r}\cdot\frac{\partial x_r}{\partial s_i} = \Sigma\xi_{i/r}\lambda_i^{(r)},$$

which, by (9), reduces to

$$C_i \equiv \omega_{ii} - \Sigma_j\gamma_{jii}\xi_j.$$

The formula shows that the invariant vector ω_{ii} is the normal component of the curvature of the curve of the congruence in question. The tangential component of the curvature is

$$\gamma = \Sigma_j\gamma_{jii}\xi_j.$$

The invariant γ is then the geodesic curvature,‡ since the curvature of the projection on the tangential space is the projection of the curvature.§ If the congruence λ_i is geodesic, $\gamma_{jii} = 0$. Hence the curvature of a geodesic is directed along the normal and is equal to ω_{ii}.

An important invariant connected with any two differential forms is

$$- \Sigma a^{(rs)}y_{rs} = \Sigma a_{rs}y^{(rs)} = \Sigma_{ijkrs}\omega_{ij}\lambda_{i/r}\lambda_{i/s}\lambda_k^{(r)}\lambda_k^{(s)} = \Sigma\omega_{kk}.$$

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* Ricci and Levi-Civita, "Méthodes de calcul différentiel absolu," etc., *Math. Annalen*, vol. 54, p. 148.
† Ricci and Levi-Civita, loc. cit., p. 148.
‡ Ricci and Levi-Civita, loc. cit., p. 154.
§ Wilson and Moore, loc. cit., p. 320.

Hence the sum of the normal curvature vectors of m mutually orthogonal curves on the spread is constant.   We shall define the mean curvature* h by the relation

$$mh = \Sigma \omega_{ii}.$$

4. *Minimal Varieties.*—Applying the preceding formulas, we can now easily derive the condition that the volume of a variety of m dimensions bounded by a variety of $m - 1$ dimensions, chosen arbitrarily, shall be a minimum.   The vector element of volume is

$$Pdx_1dx_2 \cdots dx_m = \frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} dx_1 dx_2 \cdots dx_m.$$

The condition for a minimum is

$$\delta \int (P \cdot P)^{\frac{1}{2}} dx_1 dx_2 \cdots dx_m = \int \frac{\delta P \cdot P}{(P \cdot P)^{\frac{1}{2}}} dV = \int \delta P \cdot M dV = 0,$$

where M is the unit tangent n-space

$$M = \frac{P}{(P \cdot P)^{\frac{1}{2}}}.$$

Now

$$\delta P = \frac{\partial \delta y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} + \frac{\partial y}{\partial x_1} \times \frac{\partial \delta y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m}$$

$$+ \cdots + \frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial \delta y}{\partial x_m}.$$

The condition for a minimum then becomes the vanishing of n vector integrals.   One of these integrals, after an integration by parts and the omission of the part which vanishes at both limits, becomes

$$\int \left(\frac{\partial \delta y}{\partial x_1} \times \frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} \right) \cdot M dV$$

$$= - \int \delta y \times \frac{\partial}{\partial x_1} \left(\frac{\partial y}{\partial x_2} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M \right) dV.$$

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*This is the direct generalization of mean curvature as used in the theory of surfaces.   There is, however, a mean curvature definable in terms of the first fundamental form, and this is the one used by Einstein. The definition used in this note depends on the space in which V_m lies.

The complete condition then becomes

$$\int \delta y \times \left[\frac{\partial}{\partial x_1} \left(\frac{\partial y}{\partial x_2} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M \right) - \frac{\partial}{\partial x_2} \left(\frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_3} \right. \right.$$

$$\left. \left. \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M \right) + \cdots \pm \frac{\partial}{\partial x_m} \left(\frac{\partial y}{\partial x_1} \times \cdots \times \frac{\partial y}{\partial x_{m-1}} \cdot M \right) \right] = 0.$$

Since δy is arbitrary,

$$\frac{\partial}{\partial x_1} \left(\frac{\partial y}{\partial x_2} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot M \right) - \frac{\partial}{\partial x_2} \left(\frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_3} \times \cdots \right.$$

$$\left. \times \frac{\partial y}{\partial x_m} \cdot M \right) + \cdots \pm \frac{\partial}{\partial x_m} \left(\frac{\partial y}{\partial x_1} \times \cdots \times \frac{\partial y}{\partial x_{m-1}} \cdot M \right) = 0,$$

which on multiplying out reduces to

$$\frac{\partial y}{\partial x_2} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot \frac{\partial M}{\partial x_1} - \frac{\partial y}{\partial x_1} \times \frac{\partial y}{\partial x_3} \times \cdots \times \frac{\partial y}{\partial x_m} \cdot \frac{\partial M}{\partial x_2}$$

$$+ \cdots = 0.$$

Now substituting

$$\frac{\partial y}{\partial x_r} = y_r = \Sigma_k \xi_k \lambda_{k/r},$$

$$M = \xi_1 \times \xi_2 \times \cdots \times \xi_m,$$

$$\frac{\partial M}{\partial x_i} = \frac{\partial \xi_1}{\partial x_i} \times \xi_2 \times \cdots \times \xi_m + \xi_1 \times \frac{\partial \xi_2}{\partial x_i} \times \xi_3 \times \cdots \times \xi_m + \cdots,$$

and using (9), the condition for a minimum becomes

$$(\Sigma_i \omega_{ii}) |\lambda_{i/r}| = 0.$$

The determinant $|\lambda_{i/r}|$ is equal to $\sqrt{a}$, where a is the determinant of the first fundamental form, for from the expression $a_{rs} = \Sigma_i \lambda_{i/r} \lambda_{i/s}$ we have

$$a = |a_{rs}| = |\Sigma_i \lambda_{i/r} \lambda_{i/s}| = |\lambda_{i/r}| \cdot |\lambda_{i/s}| = |\lambda_{i/r}|^2.$$

Hence the condition for a minimum is $\Sigma \omega_{ii} = 0$.

The vanishing of the mean curvature vector is a necessary and sufficient condition that the variety be minimal. This is the same condition as that for a 2-surface in hyperspace.

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