THE EINSTEIN SOLAR FIELD.

BY PROFESSOR LUTHER PFAHLER EISENHART.

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The Schwarzschild form of the linear element of the Einstein field of gravitation of a mass m at rest with respect to the space-time frame of reference is

$$(1)\qquad ds^2 = \left(1 - \frac{2m}{u_1}\right) dt^2 - \frac{u_1}{u_1 - 2m} du_1^2 - u_1^2(du_2^2 + \sin^2 u_2\, du_3^2),$$

where t is the coordinate of time, and u_1, u_2, u_3 are space coordinates. Since the coefficients in (1) are independent of t, the particle moves in the 3-space S_3 whose linear element is

$$(2)\qquad ds_0^2 = \frac{u_1}{u_1 - 2m} du_1^2 + u_1^2(du_2^2 + \sin^2 u_2\, du_3^2).$$

If we put

$$(3)\qquad \begin{aligned} x_1 &= u_1 \sin u_2 \cos u_3, & x_2 &= u_1 \sin u_2 \sin u_3, \\ x_3 &= u_1 \cos u_2, & x_4 &= 4m\sqrt{\frac{u_1}{2m} - 1}, \end{aligned}$$

we have

$$(4)\qquad ds_0^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Hence S_3 is immersed in the euclidean space of four dimensions, S_4, whose rectangular coordinates are x_i $(i = 1, \cdots, 4)$.* Moreover, as follows from (3), S_3 is the quartic variety defined by

$$(5)\qquad x_1^2 + x_2^2 + x_3^2 = \left(\frac{x_4^2}{8m} + 2m\right).$$

We shall show that equations (3) define the only three-spread with the linear element (2) in euclidean four-space by making use of the following theorem of Bianchi:† In euclidean n-space $(n > 3)$ every hypersurface is not deformable unless at least $n - 2$ of the principal radii of curvature are infinite.

* Cf. Kasner, American Journal of Mathematics, vol. 43, p. 132 (April, 1921).

† *Lezioni*, vol. 1, p. 467.

In fact we show that all of the principal radii of curvature of (5) are finite. If X_i denote the direction-cosines of the normal to (5), that is

$$\sum_{i=1}^{4} X_i \frac{\partial x_i}{\partial u_j} = 0 \ (j = 1, 2, 3), \qquad \sum X_i^2 = 1,$$

then

$$X_1 = \sqrt{\frac{2m}{u_1}} \sin u_2 \cos u_3, \qquad X_2 = \sqrt{\frac{2m}{u_1}} \sin u_2 \sin u_3,$$

$$X_3 = \sqrt{\frac{2m}{u_1}} \cos u_2, \qquad X_4 = -\sqrt{1 - \frac{2m}{u_1}}.$$

If we define functions Ω_{rs} by

$$\Omega_{rs} = -\sum_i \frac{\partial X_i}{\partial u_r} \frac{\partial x_i}{\partial u_s},$$

we find $\Omega_{rs} = 0 \ (r \neq s)$ and

$$\Omega_{11} = \frac{1}{u_1 - 2m}\sqrt{\frac{m}{2u_1}}, \quad \Omega_{22} = -\sqrt{2mu_1}, \quad \Omega_{33} = -\sqrt{2mu_1}\sin^2 u_2.$$

The principal radii of curvature are given by*

$$\frac{1}{R_1} = \frac{\Omega_{11}}{\dfrac{u_1}{u_1 - 2m}} = \sqrt{\frac{m}{2u_1^3}}, \qquad \frac{1}{R_2} = \frac{\Omega_{22}}{u_1^2} = -\sqrt{\frac{2m}{u_1^3}},$$

$$\frac{1}{R_3} = \frac{\Omega_{33}}{u_1^2 \sin^2 u_2} = \sqrt{\frac{2m}{u_1^3}},$$

which are finite since $u_1 \neq 0$.

In accordance with the Einstein theory the world-line of a particle in the gravitational field is a geodesic of the space with the linear element (1), that is a curve along which $\int ds$ is stationary; and the world-line of a ray of light is a curve for which $ds = 0$ and $\int dt$ is stationary. In each case the frame of reference can be so chosen that a particular path satisfies the condition $u_2 = \pi/2$.† From (3) it follows that for this path $x_3 = 0$, and hence the path considered by astronomers is the projection upon the plane $x_3 = 0$ of a curve on the surface

$$x_1^2 + x_2^2 = \left(\frac{x_4^2}{8m} + 2m\right)^2.$$

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* Bianchi, l.c., pp. 368, 472.

† Cf. Eddington, *Report on the Relativity Theory of Gravitation*, p. 49.

This is the surface of revolution of a parabola of latus rectum $8m$ about its directrix. A similar result was obtained by Flamm* who considered the surface, in euclidean three-space, for which the linear element is given by (2) for $u_2 = \pi/2$.

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A COVARIANT OF THREE CIRCLES.

BY PROFESSOR A. B. COBLE.

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Dr. J. L. Walsh † has stated the following theorem.

THEOREM. *If the double ratio,* $(z_1, z_3 \mid z_2, z)$, *of the four points* z_1, z_2, z_3, z *in the complex plane is a real number* λ, *then as the points* z_1, z_2, z_3 *run over the circles* C_1, C_2, C_3 (*and their interiors*) *respectively, the locus of* z *is a circle* (*and its interior*).

This locus is evidently a covariant, under the inversive group, of the three given circles, which is rational in λ. We find in (8) its equation and incidentally prove the theorem.

In conjugate coordinates $z, \bar{z}$, a circle is

$$C_1(z) = a_1 z\bar{z} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + b_1 = 0,$$

where a_1, b_1 are real, and $\alpha_1, \bar{\alpha}_1$ are conjugate imaginary. The bilinear invariant of two circles $C_1(z), C_2(z)$ is

$$[C_1, C_2] = \alpha_1\bar{\alpha}_2 + \alpha_2\bar{\alpha}_1 - a_1 b_2 - a_2 b_1.$$

It vanishes when the two circles are orthogonal. When they coincide it becomes $[C_1 C_1] = 2(\alpha_1\bar{\alpha}_1 - a_1 b_1)$. This vanishes when C_1 is a *point circle,* i.e. one whose equation is

$$(1) \qquad P_{z_i}(z) = (z - z_i)(\bar{z} - \bar{z}_i) = 0.$$

It is easily verified that

$$[C_1, P_{z_i}(z)] = -C_1(z_i); \quad [P_{z_i}(z), P_{z_k}(z)] = -P_{z_i}(z_k) = -P_{z_k}(z_i).$$

The two point circles of the pencil $C(z) + \mu K(z) = 0$ are determined by

$$[C + \mu K, C + \mu K] = [C, C] + 2\mu\, [CK] + \mu^2\, [KK] = 0.$$

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* PHYSIK. ZEITSCHR., vol. 17 (1916), p. 449.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 22 (1921), p. 101. The geometric proof of this theorem given by Dr. Walsh is very complicated. The method of proof followed here is considered by Dr. Walsh (loc. cit.,