THE SECOND MEAN VALUE THEOREM FOR SUMMABLE FUNCTIONS*

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We start with the following lemma: *If $f(x)$ is summable over the interval (a, b) and hence has an indefinite integral $F(x)$ over (a, x), $a \leqq x \leqq b$, then*

$$\lim_{h\to 0}\int_a^b \frac{f(x+h)-f(x)}{h}dx = \lim_{h\to 0}\frac{F(b+h)-F(b)}{h} - \lim_{h\to 0}\frac{F(a+h)-F(a)}{h} = f(b)-f(a),$$

provided that $f(x)$ is right (or left) continuous at the ends of the interval.

Now let $\phi(x)$ denote a monotone function right (or left) continuous at a and b, and consider the identity

$$\int_a^b \frac{F(x+h)\phi(x+h)-F(x)\phi(x)}{h}dx = (\mathrm{I}) + (\mathrm{II}),$$

where

$$(\mathrm{I}) = \int_a^b \frac{F(x+h)-F(x)}{h}\phi(x+h)dx,$$

$$(\mathrm{II}) = \int_a^b F(x)\frac{\phi(x+h)-\phi(x)}{h}dx.$$

Applying a well known theorem of Lebesgue's to (I), and the first mean value theorem to (II), since we know that the expression $[\phi(x+h)-\phi(x)]/h$ is always of the same sign when $a \leqq x \leqq b$, we have

$$F(b)\phi(b) - F(a)\phi(a) + \epsilon_h' = \int_a^b f(x)\phi(x)dx + F(\xi_h)\int_a^b \frac{\phi(x+h)-\phi(x)}{h}dx + \epsilon_h'',$$

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where ϵ_h' and ϵ_h'' vanish with h. Applying the lemma to the last integral, we have:

$$(1)\quad F(b)\phi(b) - F(a)\phi(a) = \int_a^b f(x)\phi(x)dx + F(\xi_h)(\phi(b) - \phi(a)) + \epsilon_h,$$

where $\lim_{h\to 0} \epsilon_h = 0$.

This can be written in the form

$$\int_a^b f(x)\phi(x) = \phi(b)\int_a^{\xi_h} f(x) + \phi(a)\int_{\xi_h}^b f(x) + \epsilon_h.$$

When ϵ_h takes on the value zero, $F(\xi_h)$, which is continuous, will take on for some value ξ, lying between a and b, a value such that

$$\int_a^b f(x)\phi(x) = \phi(b)\int_a^{\xi} f(x) + \phi(a)\int_{\xi}^b f(x),$$

which is Weierstrass's form of the second mean value theorem for integrals.

If, for example, we suppose $\phi(x)$ increasing (monotonically) and replace $\phi(x)$ by A (fixed) over the interval $(a, a+k)$ and by B (fixed) over the interval $(b, b+k)$, leaving the values of $\phi(x)$ unchanged over $(a+k, b)$, we can prove in the same way that

$$\int_a^b f(x)\phi(x) = A\int_a^{\xi} f(x) + B\int_{\xi}^b f(x),$$

where

$$A \leqq \phi(x) \leqq B$$

over (a, b), by applying the same reasoning and letting the parameter k approach zero.

From this form the usual Bonnet forms of the theorem are at once deducible.

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