ANALOGIES BETWEEN THE u_n, v_n OF LUCAS AND ELLIPTIC FUNCTIONS*

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1. *Historical Note.* The u_n, v_n of Lucas are the symmetric functions (n any real integer)

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n,$$

of the roots α, β of $x^2 = px - q$, where p, q are relatively prime integers, so that

$$(u_0, u_1) = (0, 1), \qquad (v_0, v_1) = (2, p),$$

and u_n, v_n are integers satisfying the recurrence

$$x_{n+2} = px_{n+1} - qx_n.$$

The numerous remarkable properties and applications of these integers due to Lucas and others are summarized in vol. 1, chap. XVII of Dickson's *History.* From another source it is known that as early as 1878 Lucas had applied principles similar to those of his fundamental memoir † to symmetric functions of the roots of any algebraic equation and that he had obtained the connection of these, through the intermediary of elliptic and abelian functions, with the theory of numbers. This connection is still to be sought. In 1912 the writer was informed by the late C. A. Laisant, at one time a trustee of Lucas' manuscripts, that there was nowhere in them a vestige of the subject. Nevertheless Laisant recalled vividly that Lucas, about 1878, made a verbal communication to the Société Mathématique de France in which he exhibited a close isomorphism between three symmetric functions, of which one was $\alpha^n + \beta^n + \gamma^n$, of the roots α, β, γ of a cubic equation and the elliptic functions sn, cn, dn, especially as regards a species of double periodicity. All traces of this communica-

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tion, also the sense in which the symmetric functions were doubly periodic, have been lost.

Laisant also is authority for the statement that the ultimate object of Lucas in his researches on recurring series was the demonstration of Fermat's last theorem. He recalled that in 1890, shortly before Lucas' death, the latter, "in less than a quarter of an hour," reduced the proof of Fermat's theorem to the problem of showing that each of his symmetric functions had not more than two "periods." Laisant remarked that at the time he perfectly followed this reduction, which was clear and convincing. This adds another puzzle to the many already surrounding Fermat's theorem.

In his memoir Lucas insists strongly on the isomorphism between his u_n, v_n when $q = 1$ and the circular and hyperbolic functions. Later (*Théorie des Nombres,* p. 319) he obtains this isomorphism by a simpler method, deriving the initial correspondence directly from Simpson's recurrences for the sine and cosine. This particular device does not seem to be capable of extension. The significant fact, however, apparently is that the correspondence thus established is between the *multiplication* theory of the circular or hyperbolic functions and u_n, v_n. For on page 203 of his memoir he concludes a section with "the demonstration of formulas of extreme importance; for they will serve us later as the base of the theory of doubly periodic numerical functions deduced from the consideration of symmetric functions of the roots of equations of the third and fourth degrees with commensurable coefficients." These are the formulas (A), (A') below. We shall see that (A') is incorrect, but that (A), which is correct, corresponds to classical formulas in the real multiplication of elliptic functions and to recent results in the theory of complex multiplication. The problem seems to narrow down to connecting (A) in some simple manner with the symmetric function $s_n \equiv \alpha^n + \beta^n + \gamma^n$, for any solution of $x_{n+3} = px_{n+2} - qx_{n+1} + rx_n$ is of the form $a_0 s_n + a_1 s_{n+1} + a_2 s_{n+2}$, where α, β, γ are the roots of $x^3 = px^2 - qx + r$. It may be worth noting that some of the elliptic functions that occur in the analogies are precisely those which present themselves in the conversion of the square root of a polynomial of the third or fourth degree in x into a continued fraction. In this connection we may also call attention to the remarks of Lucas at the end of § X, *Théorie des Nombres,* p. 508. Possibly the analogies noted below may offer a clue to what Lucas had in mind.

2. *Analogies with Elliptic Functions.* Having proved the formula

$$(A)\qquad u_n^2 u_{m-1}u_{m+1} - u_m^2 u_{n-1}u_{n+1} = q^{n-1}u_{m-n}u_{m+n},$$

Lucas states that in the same way we have

$$(A')\qquad v_n^2 v_{m-1}v_{m+1} - v_m^2 v_{n-1}v_{n+1} = -\Delta q^{n-1}v_{m-n}v_{m+n},$$

where $\Delta = (\alpha - \beta)^2 = p^2 - 4q$, and he puts $m = n + 1$, $n + 2$ in (A), getting

$$(B)\qquad \begin{aligned} u_n^3 u_{n+2} - u_{n+1}^3 u_{n-1} &= q^{n-1}u_1u_{2n+1}, \\ u_n^2 u_{n+1}u_{n+3} - u_{n+2}^2 u_{n-1}u_{n+1} &= q^{n-1}u_2u_{2n+2}. \end{aligned}$$

He states that "the formulas (A) and (B) belong to the theory of elliptic functions, and, more especially, to the functions which Jacobi has denoted by Θ, H."

To see that (A') is incorrect we need consider only the case in which $q = 1$, so that $v_{-n} = v_n$. If m, n be interchanged in (A') the left changes sign while the right does not. Instead of (A') we should have

$$(A'')\qquad v_n^2 v_{m-1}v_{m+1} - v_m^2 v_{n-1}v_{n+1} = -\Delta^2 q^{n-1}u_{m-n}u_{m+n}.$$

To obtain the analogies with elliptic functions let

$$(1)\qquad I(a - b,\ a - c,\ a - d,\ \cdots) = 0$$

be an identity between the differences of $a, b, c, d, \cdots$, and define a function $f_{x,\,y}$ of two variables x, y by

$$(2)\qquad f_{x,\,y} \equiv k(f_{x,\,l} - f_{y,\,l}),$$

where $k \neq 0$ and l are arbitrary. Then (1) implies that

$$I\left(\frac{1}{k}f_{a,b},\ \frac{1}{k}f_{a,c},\ \frac{1}{k}f_{a,d},\ \cdots\right) = 0;$$

and, in particular, from

$$(a-b)(c-d)+(b-c)(a-d)+(c-a)(b-d)=0,$$

we infer

$$(3) \qquad f_{a,b}f_{c,d}+f_{b,c}f_{a,d}+f_{c,a}f_{b,d}=0.$$

If now there exist functions g_x, h_x such that

$$(4) \qquad f_{x,y}=r\frac{g_{x+y}g_{x-y}}{h_xh_y},$$

where r is independent of x, y, we have $g_{-x}=-g_x$, since $f_{x,y}=-f_{y,x}$, and (3) becomes

$$(5) \quad g_{a+b}g_{a-b}g_{c+d}g_{c-d}+g_{b+c}g_{b-c}g_{a+d}g_{a-d}+g_{c+a}g_{c-a}g_{b+d}g_{b-d}=0.$$

From its derivation (5) is a consequence of (2) and (4) alone; that is, (5) is implied by

$$(6) \qquad \frac{g_{x-l}g_{x+l}}{h_xh_l}-\frac{g_{y-l}g_{y+l}}{h_yh_l}=\frac{g_{x+y}g_{x-y}}{kh_xh_y},$$

and in particular (5) is implied by

$$(7) \qquad g_y^2g_{x-1}g_{x+1}-g_x^2g_{y-1}g_{y+1}=g_1^2g_{x+y}g_{x-y},$$

which is obtained from (6) by putting $k=l=1$, $h_x=g_x^2$. Conversely, (5) implies (7), for (5) becomes (7) when

$$(a,b,c,d)=(1,x,y,0).$$

From a given solution $g_x\equiv p_x$ of (5) and (7) we can obtain several more by devices familiar in the multiplication theory of elliptic functions. Thus, since the sum of the squares of the suffixes of the g's in (5) is $2(a^2+b^2+c^2+d^2)$ for each term,

$$g_x\equiv p_x/h_l^{x^2}$$

is also a solution of (5) and (7).

Henceforth, as in establishing the correspondence between u_n, v_n and the circular or hyperbolic functions, we take $q=1$, so that now

$$(8) \qquad u_{-n}=-u_n, \qquad v_{-n}=v_n.$$

From the second of (8), it follows at once that $g_x\equiv v_x$ is not a solution of (5). If (A') were correct the contrary would be the case.

Let a, b, c, d, m, n, x, y denote real integers. Then $g_x \equiv u_x$ is a solution of (7), as is seen from (A), and hence also of (5), since $u_1 = 1$. To preserve the double homogeneity of (5) and its consequences we retain all powers of u_1 arising from special choices of a, b, c, d. Thus the formulas (B) should be written $(q = 1)$,

$$(B') \qquad \begin{aligned} u_n^3 u_{n+2} - u_{n+1}^3 u_{n-1} &= u_1^3 u_{2n+1}, \\ u_n^2 u_{n+1} u_{n+3} - u_n^2 u_{n-1} u_{n+1} &= u_1 u_2 u_{2n+2}. \end{aligned}$$

Write Jacobi's $\mathrm{H}(z) \equiv \mathrm{H}$, $\mathrm{H}(nz) \equiv \mathrm{H}_n$ $(n \neq 1)$. Then for $g_x \equiv \mathrm{H}_x$, (5) becomes a well known form of Jacobi's theta elation, so that H_x is a solution of (7). This verifies Lucas' statement that (A), (B) belong to the theory of the H function (with $q = 1$ and the above modification (B') of (B)). A somewhat closer analogy is given by the solutions

$$g_x \equiv \eta_x \equiv \mathrm{H}_x/\mathrm{H}^{x^2}, \qquad g_x \equiv \zeta_x \equiv \mathrm{H}_x \mathrm{H}^{(x^2-4)/4}/\mathrm{H}_2^{(x^2-1)/3}$$

of (5), ζ_x being the familiar elliptic function occurring in the theory of Poncelet polygons. In these cases, analogously to $(u_0, u_1) = (0, 1)$, we have

$$(\eta_0, \eta_1) = (\zeta_0, \zeta_1) = (0, 1).$$

Analogies with the Weierstrassian

$$\sigma(nz) \equiv \sigma_n, \quad \wp(nz) \equiv \wp_n$$

are obtained by means of the well known

$$\psi_n(z) \equiv \psi_n = \sigma_n/\sigma_1^{n^2},$$

which is fundamental in the theory of real multiplication. For $g_x \equiv \psi_x$ is a solution of (5), and $(\psi_0, \psi_1) \equiv (0, 1)$. Other solutions are Halphen's

$$\gamma_x \equiv \psi_x/\psi_2^{(x^2-1)/3}, \qquad \delta_x \equiv \gamma_{mx}\gamma_m^{(x^2-4)/3}/\gamma_{2m}^{(x^2-1)/3},$$

for which $(\gamma_0, \gamma_1, \gamma_2) = (\delta_0, \delta_1, \delta_2) = (0, 1, 1)$. In transposing formulas relating to γ, δ into terms of u, all powers of δ_2, γ_2

must be retained, since $u_2 = 1$ (when $q = 1$) only when $p = 1$. The solution $g_x \equiv \sigma_x$ of (5), giving Weierstrass' equation of three terms, yields analogies between u_x and σ_x. The function γ_x is that occurring in the conversion of the square root of a polynomial of the third or fourth degree into a continued fraction; δ_x is used in abridging the computation of γ_{mn}. In this last there is a resemblance to the process by which Lucas converts his u_x, v_x formulas into others concerning u_{mx}, v_{mx}. The usual methods for calculating γ_x as a polynomial in γ_3^3, γ_4 can evidently be carried over bodily to the computation of u_x. More generally, we see from the foregoing solutions of (5) that any relation between any one of H_x, η_x, ζ_x, σ_x, ψ_x, γ_x, δ_x for different ranks x can be transposed into a relation between u_x's of the same ranks. The converse does not hold; only such properties of u_x are translatable into terms of H_x, $\cdots$, δ_x as can be obtained from (5) alone. A correspondence between $\wp$'s and u's is established by means of the functions $-\wp_{x,\,y}$, $u_{x,\,y}$ of two arguments x, y (of which y does not assume the value zero),

$$\wp_{x,\,y} \equiv \wp_x - \wp_y = \wp_{x,\,1} - \wp_{y,\,1} = -\frac{\psi_{x+y}\psi_{x-y}}{\psi_x^2\psi_y^2},$$

$$u_{x,\,y} \equiv \frac{u_{x+y}u_{x-y}}{u_x^2 u_y^2} = u_{x,\,1} - u_{y,\,1}.$$

Although Lucas did not consider u_x when x is a complex number, we see at once that (5) is valid when $q = 1$, a, b, c, d are complex, and $g_x \equiv u_x$. It has recently been shown by Berwick * that the same relation (5) holds for certain functions ψ upon which the complex multiplication of $\wp(z)$ depends. Thus u_x has analogies in both the real and the complex multiplication of elliptic functions.

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* PROCEEDINGS OF THE LONDON SOCIETY, vol. 19 (1920), p. 153.