from (16), on eliminating h or k, that $h = \mu h_1$, $k = \lambda k_1$, where h_1 and k_1 are integral, and we get

(17) $h_1 a_1 + k_1 b_1 \equiv 0 \pmod{\varepsilon_1}$, $h_1 c_1 + k_1 d_1 \equiv 0 \pmod{\varepsilon_1}$.

The nature of the singularities on the sides of the triangle ABC is readily determined. For instance, suppose in (6) $c > a > 0$. Then (6) gives an expansion for t in ascending powers of $x^{1/a}$, and thence we get for y an expansion of the form

$$y = x^{c/a}(\alpha + \beta x^{1/a} + \gamma x^{2/a} + \cdots)$$

in general, fixing the nature of the singularity for which t is zero.

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SURFACES WITH ORTHOGONAL LOCI OF THE CENTERS OF GEODESIC CURVATURE OF AN ORTHOGONAL SYSTEM*

BY MALCOLM FOSTER

We consider a surface S referred to any orthogonal system. Let G_1 and G_2 be the centers of geodesic curvature of the curves u = const. and v = const. respectively, through any point M of S. As M is displaced over the entire surface the loci of G_1 and G_2 will in general be two surfaces S_1 and S_2, corresponding elements of which are those which result from a common displacement of M. We ask: What are the surfaces S for which the surfaces S_1 and S_2 correspond with orthogonality of linear elements?

The condition that the displacements of G_1 and G_2 be orthogonal for every displacement of M, is that the absolute displacements of these points in the directions of the axes of the moving trihedral at M satisfy the relation

$$\sum \delta x_1 \, \delta x_2 = 0, \tag{1}$$

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for all values of dv/du. The x-axis of the trihedral is chosen tangent to the curve $v =$ const. The radii of geodesic curvature of the curves $u =$ const. and $v =$ const. we denote by ϱ_{gv} and ϱ_{gu} respectively. The relation (1) becomes*

$$\varrho_{gu} r_1 \left(-\frac{\partial \varrho_{gv}}{\partial u} du - \frac{\partial \varrho_{gv}}{\partial v} dv + \xi du\right) dv$$
$$+ \varrho_{gv} r \left(\frac{\partial \varrho_{gu}}{\partial u} du + \frac{\partial \varrho_{gu}}{\partial v} dv + \eta_1 dv\right) du$$
$$- \varrho_{gv} \varrho_{gu} (p du + p_1 dv)(q du + q_1 dv) \equiv 0.$$

Hence, setting the coefficients of du^2, $du dv$, and dv^2 equal to zero, we get

$$(2) \quad \begin{cases} \varrho_{gv}\left(\varrho_{gu} p q - r \dfrac{\partial \varrho_{gu}}{du}\right) = 0, \\ \varrho_{gu} r_1 \left(\xi - \dfrac{\partial \varrho_{gv}}{\partial u}\right) + \varrho_{gv} r \left(\eta_1 + \dfrac{\partial \varrho_{gu}}{\partial v}\right) \\ \qquad - \varrho_{gv} \varrho_{gu} (p q_1 + p_1 q) = 0, \\ \varrho_{gu}\left(\varrho_{gv} p_1 q_1 + r_1 \dfrac{\partial \varrho_{gv}}{\partial v}\right) = 0. \end{cases}$$

Now $\varrho_{gv} = \dfrac{\eta_1}{r_1}$, and $\varrho_{gu} = \dfrac{\xi}{r}$;† using these values, and the relations between the fundamental quantities for the surface‡, the equations (2) reduce to

$$(3) \quad \begin{cases} \varrho_{gu} p q - r \dfrac{\partial \varrho_{gu}}{\partial u} = 0, \\ p q_1 = 0, \\ \varrho_{gv} p_1 q_1 + r_1 \dfrac{\partial \varrho_{gv}}{\partial v} = 0, \end{cases}$$

since $\varrho_{gv}, \varrho_{gu} \neq 0$.

Since $p = D'/\eta$, and $q_1 = - D'/\xi$,§ we see from the second member of (3) that both p and q_1, are zero, and that the parametric curves must be the lines of curvature.

* Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 170.

† Eisenhart, p. 132, formula (47), and p. 167, formulas (45).

‡ Eisenhart, p. 168 and p. 170.

§ Eisenhart, p. 174, formulas (73).

Consequently the first and third members of (3) reduce to*

$$\frac{\partial \varrho_{gu}}{\partial u} = 0,$$

$$\frac{\partial \varrho_{gv}}{\partial v} = 0.$$

Hence

$$(4) \qquad \varrho_{gv} = U, \qquad \varrho_{gu} = V,$$

where U and V are functions of u and v alone respectively. The parametric curves therefore have constant geodesic curvature and the system is isothermal.† The surface is therefore isothermal.

Making use of (4), we see that the elements ds_1^2 and ds_2^2 of the loci of G_1 and G_2 respectively, are

$$(5) \quad \begin{cases} ds_1^2 = \sum \delta x_1^2 = \dfrac{\eta_1^2}{r_1^2}\left[\dfrac{\left(\dfrac{\partial r_1}{\partial u}\right)^2}{r_1^2} + r^2 + q^2\right] du^2, \\ ds_2^2 = \sum \delta x_2^2 = \dfrac{\xi^2}{r^2}\left[\dfrac{\left(\dfrac{\partial r}{\delta v}\right)^2}{r^2} + r_1^2 + p_1^2\right] dv^2. \end{cases}$$

Hence the loci of G_1 and G_2 are curves and not surfaces. As the vertex of the trihedral describes a curve $u =$ const. the point G_1 remains fixed, and as the vertex of the trihedral describes a curve $v =$ const. the point G_2 remains fixed. The lines of curvature are therefore spherical in both systems; they lie on spheres whose centers lie on the loci of G_1 and G_2, and which are mutually orthogonal with S at every point.

We denote the curves which are the loci of G_1 and G_2 by Γ_1 and Γ_2 respectively. The curve Γ_1 is described by G_1 as the vertex of the trihedral describes every curve $v =$ const., and Γ_2 is described by G_2 as the vertex of

* We exclude the cases where either $r = 0$, or $r_1 = 0$, since in either case the curves in one family are geodesics, and one of the points G_1 and G_2 is at infinity.

† Eisenhart, p. 137.

the trihedral describes every curve $u = \text{const.}$ Suppose that the vertex of the trihedral describes a definite curve $u = \text{const.}$ The point G_1 remains fixed at some point on Γ_1 while G_2 describes Γ_2, and every tangent to Γ_2 will be perpendicular to the fixed tangent to Γ_1 at G_1. The curve Γ_2 is therefore either a plane curve whose plane is perpendicular to the fixed tangent to Γ_1, or a straight line perpendicular to this fixed tangent. Suppose now that the vertex of the trihedral describe a second curve $u = \text{const.}$ Then G_1 remains fixed at some second point of Γ_1 while G_2 describes Γ_2, and every tangent to Γ_2 will be perpendicular to the fixed tangent to Γ_1 at the second position of G_1. Consequently if Γ_2 be a plane curve, the locus of G_1, namely Γ_1, must be a straight line. We obtain similar conclusions if we consider the vertex of the trihedral to describe two different curves $v = \text{const.}$ Hence the locus of at least one of the points G_1 and G_2 is a straight line.

We suppose that it is the locus of G_1 which is a straight line. Consider the absolute displacements of the point G_1 in the directions of the axes of the trihedral T_u of a curve $v = \text{const.}$ We have*

$$\frac{\delta x_1}{ds} = \frac{dx_1}{ds} + 1, \quad \frac{\delta y_1}{ds} = \frac{x_1}{\varrho}, \quad \frac{\delta z_1}{ds} = 0, \tag{6}$$

where ds is the element of arc of the curve $v = \text{const.}$, and ϱ is the radius of first curvature. From the third member of (6), we see that the line Γ_1 which is the locus of G_1, lies in the osculating plane of the curve $v = \text{const.}$ at every point. Now

$$x_1 = -\varrho_{gv} = -\frac{\eta_1}{r_1},\dagger \qquad ds = \sqrt{E}\,du = \xi du;$$

using these relations, together with the relations between the fundamental quantities for the surface,‡ equations (6) become

* Eisenhart, p. 32.

† It is necessary tha ρ_{gv} be measured in the opposite direction to that in which the parameter u increases. Cf. Darboux, vol. II, p. 359.

‡ Eisenhart, p. 168 and p. 170.

$$(7) \qquad \frac{\delta x_1}{ds} = \frac{\eta_1 \dfrac{\partial r_1}{\partial u}}{\xi r_1^2}, \quad \frac{\delta y_1}{ds} = -\frac{\eta_1}{\varrho r_1}, \quad \frac{\delta z_1}{ds} = 0.$$

Hence

$$(8) \qquad ds_1 = \frac{\eta_1 \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{1/2} ds}{\varrho \xi r_1^2},$$

where ds_1 is the element of the line Γ_1. The direction-cosines of Γ_1 relative to the trihedral T_u are therefore

$$(9) \; \alpha_1 = \frac{\varrho \dfrac{\partial r_1}{\partial u}}{\sqrt{\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2}}, \; \beta_1 = \frac{-\xi r_1}{\sqrt{\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2}}, \; \gamma_1 = 0.$$

Since Γ_1 is a straight line fixed in space, we must have

$$\frac{\delta \alpha_1}{ds} = \frac{\delta \beta_1}{ds} = \frac{\delta \gamma_1}{ds} = 0.$$

These equations become on using (9),*

$$(10) \left\{ \begin{aligned} &\frac{\delta \alpha_1}{ds} = \frac{d\alpha_1}{ds} - \frac{\beta_1}{\varrho} \\ &= \frac{r_1^2 \left[\varrho \xi \left(\varrho \dfrac{\partial^2 r_1}{\partial u^2} + \dfrac{\partial \varrho}{\partial u} \dfrac{\partial r_1}{\partial u}\right) - \varrho^2 \dfrac{\partial r_1}{\partial u} \dfrac{\partial \xi}{\partial u} + \xi^3 r_1\right]}{\varrho \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{3/2}} = 0, \\ &\frac{\delta \beta_1}{ds} = \frac{d\beta_1}{ds} + \frac{\alpha_1}{\varrho} \\ &= \frac{r_1 \dfrac{\partial r_1}{\partial u}\left[\varrho \xi \left(\varrho \dfrac{\partial^2 r_1}{\partial u^2} + \dfrac{\partial \varrho}{\partial u} \dfrac{\partial r_1}{\partial u}\right) - \varrho^2 \dfrac{\partial r_1}{\partial u} \dfrac{\partial \xi}{\partial u} + \xi^3 r_1\right]}{\xi \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{3/2}} = 0, \\ &\frac{\delta \gamma_1}{ds} = -\frac{\beta_1}{\tau} = \frac{\xi \eta_1 r_1}{\tau \left[\varrho^2 \left(\dfrac{\partial r_1}{\partial u}\right)^2 + \xi^2 r_1^2\right]^{1/2}} = 0, \end{aligned} \right.$$

* Eisenhart, p. 32.

where τ is the radius of second curvature of the curve $v = \text{const.}$ Hence we must have

$$
(11) \qquad \begin{cases} \varrho\xi\left(\varrho\frac{\partial^2 r_1}{\partial u^2} + \frac{\partial\varrho}{\partial u}\frac{\partial r_1}{\partial u}\right) - \varrho^2\frac{\partial r_1}{\partial u}\frac{\partial\xi}{\partial u} + \xi^3 r_1 = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{\tau} = 0. \end{cases}
$$

The curves $v = \text{const.}$ are therefore plane; and since they are spherical, they are circles. Consequently $\varrho = \text{const.}$, and the first member of (11) reduces to

$$
(12) \qquad \varrho^2\left(\xi\frac{\partial^2 r_1}{\partial u^2} - \frac{\partial r_1}{\partial u}\frac{\partial\xi}{\partial u}\right) + \xi^3 r_1 = 0.
$$

The relation (12) may also be written in the form

$$
(13) \qquad \varrho^2\frac{\partial}{\partial u}\left(\frac{\frac{\partial r_1}{\partial u}}{\xi}\right) + \frac{\partial\eta_1}{\partial u} = 0.
$$

The line Γ_1 therefore lies in the plane of every curve $v = \text{const.}$, and consequently the surfaces have plane lines of curvature in one system for which all the planes pass through the straight line Γ_1; such surfaces are called surfaces of Joachimsthal.*

Finally, the surfaces considered are isothermal surfaces of Joachimsthal for which the lines of curvature which lie in coaxial planes are circles, and for which either the relation (12) holds, or the corresponding relation is satisfied with reference to the curves $u = \text{const.}$

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* Eisenhart, pp. 308–310.