VEBLEN ON ANALYSIS SITUS

The Cambridge Colloquium, 1916, Part II. *Analysis Situs.* By Oswald Veblen. New York, The American Mathematical Society, 1922. vii + 150 pp.

The present reviewer finds himself facing the infrequent experience of tackling the same work for the second time. That he derives no little satisfaction from it, is due to the fact that since the first review was written* he has made ample use of the book as an instrument of research, and hopes that the additional experience thus acquired will prove of value not only to himself but also to his readers.

Let us say at once that the *Lectures* have stood the test of usage very well indeed. Hitherto a beginner attracted by the fascinating and difficult field of analysis situs, was obliged to wade through many widely scattered papers, beginning with Poincaré's classic of the 1895 in JOURNAL DE L'ÉCOLE POLYTECHNIQUE and its multiple *Compléments.* Difficult reasonings beset him at every step, an unfriendly notation did not help matters, to all of which must be added, most baffling of all, the breakdown of geometric intuition precisely when most needed. No royal road can be created through this dense forest, but a good and thoroughgoing treatment of fundamentals, notation, terminology, may smooth the path somewhat. And this and much more we find supplied by Veblen's *Lectures.* That few, if any, were better qualified than the author, by temperament and scientific past, to produce such a work, is well known among the devotees of analysis situs, whose number it will assuredly increase. As this field presents difficult problems in profusion, with but few general methods, such an increment is very much to be hoped for.

Two streams are found in analysis situs: the one related to point-set theory, the other primarily of a combinatorial nature, wherein are treated manifolds which for example in the case of two dimensions, are depictable upon polyhedra with a finite number of plane polygonal faces. To this latter branch these *Colloquium Lectures* are almost wholly devoted. Withal the author is perhaps at his mathematical best at the points of contact of the two streams, say with the type of question presented by the occurrence of Jordan curves on a representative polyhedron.

Veblen has made extensive and very systematic use of the Poincaré

* See *Bulletin des Sciences Mathématiques* for 1922.

† A k-cell is any point-set continuously depictable upon the interior of a hypersphere in k-space.

matrices of a manifold, whereby the incidence relations beween the k-cells and $(k-1)$-cells† are defined by the elements of a certain matrix. These elements are $0, \pm 1$, when the orientations of the cells are taken into consideration. In a notable paper (ANNALS OF MATHEMATICS, 1913) Veblen and Alexander showed that for many purposes the elements could be taken merely modulo 2. By comparing all these matrices and reducing them in the usual manner, one obtains the all important *topological invariants* and their relations. Our geometer friends will ruefully complain that the structure is much too analytical for their taste, but there is no help to it for the present, as nothing else comparably rigorous is now in print. To the above aspect of the theory are devoted four of the five Lectures. Of particular excellence are the elegant development of normal forms for two-dimensional manifolds, the treatment of questions of orientation, also of *covering complexes*,—one of the most useful features of the book. Everything pertaining to general manifolds immersed in a given one with assigned cellular structure is thereby thoroughly clarified. Of decided interest in this connection is the author's reduction of arbitrary k-circuits (k-cycles) on a manifold to those composed exclusively of k-cells of a particular cellular subdivision. Thus a Jordan curve on a polyhedron would be reduced to a polygon of edges. From this follows indeed that certain integers related to the representative matrices (Betti numbers, torsion-coefficients, etc.) are actually topological invariants.

The fifth and last Lecture differs in type from the others, and is more in the nature of what we have been accustomed to expect of such Lectures. It contains an excellent summary of several important questions: homotopy and isotopy, theory of the indicatrix, a fairly ample treatment of the group of a manifold, finally a bird's eye view of what is known and not known (mostly the latter) on three dimensional manifolds.

There is no fundamental criticism to level at the *Lectures*. An error in the argument pp. 142-143 has been pointed out by J. Nielsen of Copenhagen. And if one has some good important theorems, why refrain from displaying them appropriately by means of heavy type, titles, and the like? Of this sin the author is repeatedly guilty: it does not facilitate the beginner's task. We should also have been most thankful for a more complete bibliography.

These *Colloquium Lectures*, one of the best productions of a Series of which American mathematics may well be proud, deserve to be and will be well received by fellow workers in this field and by the general mathematical public.

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