BOREL'S THEORY OF FUNCTIONS.

Méthodes et Problèmes de la Théorie des Fonctions. By Émile Borel. (Collection de monographies sur la théorie des fonctions.) Paris, Gautier-Villars, 1922. ix + 148 pp.

The volume under review, which Borel states in the introduction is to be the last of this justly celebrated collection of monographs to appear under his own name, is designed to supplement the earlier volumes of the series by bringing together in their original form a number of his notes and memoirs on the theory of functions. The twenty-seven articles included in this volume appeared in a number of journals during the years 1895—1912. They cover a wide range of subject matter and indicate clearly the breadth, originality, and versatility which have been the source of the very great influence which Borel has exerted on the progress of analysis in the last thirty years.

The memoirs presented are grouped in four chapters: *Les domaines et la théorie des ensembles*; *Les opérations et les développements en séries*; *La théorie de la croissance et le rôle des constants arbitraires*; *Les fonctions de variable complexe, en général, et les fonctions particulières.*

The introduction is devoted to an interesting analysis of the many striking analogies between biology and the theory of functions, both in content and in historical order of development. As man learned to use animals and plants before he had acquired a thoroughgoing knowledge of anatomy or physiology so mathematicians discovered and utilized the elementary functions, algebraic, circular, exponential. In biology the study of the structure of organisms led to the analysis of cellular life and organic chemistry; in the theory of functions the analysis of the properties of functions brought into existence the modern theories of aggregates, numbers, sets of points.

In the first chapter Borel remarks that the domains and ensembles are to functions as the tissues are to living organisms and calls attention to the importance which this part of the theory of functions has come to assume. Of the seven papers included in this chapter three are brief notes, one on the representation of discontinuous functions as limits of continuous functions and two on the theory of measure. The last of these outlines a procedure which may be substituted for the method of Lebesgue in the theory of integration. Three articles are devoted to the analysis and classification of sets of measure null. It is significant that the discoverer of the role in analysis of the sets of measure null should be the pioneer in the study of their structure and the closely related theory of monogenic non-analytic functions.

This chapter also contains a memoir entitled, *"Sur les définitions analytiques et sur l'illusion du transfini"*, which presents ably and forcibly the author's views regarding the limitations which should be placed upon arguments involving the transfinite. As is well known, no finite collection of words and symbols is adequate to the representation of all the transfinite ordinals of the second class. Borel calls attention to the relation between these ordinals α and the corresponding types of increasing function $\varphi_\alpha(n)$. In terms of these functions $\varphi_\alpha(n)$ it is possible to define a function $f(x)$ which is not representable analytically, but the complete definition of the function $f(x)$ rests upon the assumed definition of the $\varphi_\alpha(n)$ and the latter definition is impossible in a finite number of words.

The second chapter, on operations and developments in series, contains a short memoir on the conditions governing a change in the order of the terms of a conditionally convergent series, a memoir on the representation of functions of two real variables, and one on the integration of unbounded functions and constructive definitions. In his discussion of constructive definitions Borel expounds in detail his view that the only entities which may properly enter a mathematical discussion are those which are "well-defined". The continuum of Cantor and Dedekind is not "well-defined" in the sense of Borel since the numbers which form the continuum are not all definitely given. Hence, to be entirely consistent, it would be necessary for Borel either to abandon the use of the concept of the continuum and the related theory of functions or else so to transform these theories that they rest upon a "well-defined" basis. In spite of the admonishments of Borel and other adherents of his views, the number of those who call themselves mathematicians and who devote themselves to the study of concepts which are not "well-defined", such as non-measurable functions, transfinite cardinals and ordinals, measurable sets which are not Borel sets, curves in non-metrical analysis situs, is large and is increasing. It is not possible for any one group to guide or stem the tide of mathematical investigation or to say that here is mathematics; thus far and no farther may it go.

In the preliminary remarks to the third chapter Borel points out that in biology the greater interest and value lies in the study of the existent and normal species rather than in the study of hypothetical non-existent or abnormal species. By analogy, in mathematics the subject of functions which increase exponentially (to which that chapter is devoted) is relatively of great importance. The chapter contains ten notes, of which three are on the theory of the increase of integral functions and three are on various modes of approximation. One of the remaining four notes is concerned with an analogy between the number e and the algebraic numbers, one with an analytic partial differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} - a^4 \frac{\partial^2 \varphi}{\partial y^2} = \psi (x, y)$$

which has for certain values of a a solution $\varphi (x, y)$ which is nowhere analytic in x or y; another to a formula for any analytic integral of a linear partial differential equation with analytic coefficients, and the last to the periods of abelian integrals and a related generalization.

The fourth chapter contains seven notes on the theory of functions of a complex variable which appeared in the COMPTES RENDUS from 1897—1912. If $\varphi (z)$ is a function with zeros at $a_1, a_2, \ldots$, the formula

$$f(z) = \sum \frac{c_n \varphi (z)}{(z - a_n) \varphi' (a_n)}$$

determines, in the case of convergence, a function $f(z)$ which assumes prescribed values c_n for $z = a_n$. In the first note Borel considers conditions that φ and so f be unique and brings out the intimate relationship between the theory of the zeros of integral functions and the problem of determining a function $f(x)$ uniquely by an enumerable infinity of conditions. The next three notes consider the relations between the coefficients of a power series and the singularities of the corresponding function. The fifth note, entitled *Sur l'étude asymptotique des fonctions méromorphes* (1902), is of historical interest. In this note Borel applied the method of exclusion of singularities to the case of functions whose singularities are everywhere dense in a region and was thereby led by a natural sequence of ideas to the discovery of the importance of the sets of measure null. The sixth and seventh notes are concerned respectively with a remarkable and unexplained relation between the coefficients of certain differential equations and the invariants of binary forms and with the variation of an analytic function in the neighborhood of an essential singularity.

In a brief conclusion Borel recommends that mathematicians seek to classify systematically those mathematical entities which are well-defined. He shows clearly that this work will rest upon a profound study and classification of those incommensurable numbers which are truly defineable, and remarks that although the subject is very difficult it is one in which the least conquest is infinitely precious because of its reactions; one attains here the living substance itself of which all mathematical entities are made; nothing is more important than the properties of number.

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