ON A GENERALIZATION OF THE SECULAR EQUATION*

BY JAMES PIERPONT

1 *Introduction.* The equation we wish to consider is

$$(1) \qquad H_n(x) = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} = 0.$$

Here $a_{ij} = a_{ji}$, when $i \neq j$; while

$$a_{ii} = \alpha_{ii} - x, \quad \text{for } i = 1, 2, \cdots, r,$$
$$= \alpha_{ii} + x, \quad \text{for } i = r+1, r+2, \cdots, n.$$

The a_{ij} and α_{ii} are real, and $H(0) \neq 0$. If $r = n$, (1) is the secular equation which it will be convenient to denote by $L_n(x) = 0$. When $r = n - 1$ the equation (1) plays a fundamental role in classifying quadric surfaces in n-way hyperbolic space. Let us set $n - r = s$ and call $\sigma = |r - s|$ the *signature* of (1). We have then the

THEOREM I. *The number of real roots of $H_n(x) = 0$, counting their multiplicity, is not less than its signature.*

This is a corollary of a theorem to which F. Klein calls especial attention (Mathematische Annalen, vol. 23 (1884), p. 562). The proof there given rests on the theory of elementary divisors.† We give here a very simple proof which is a modification of H. Weber's proof that the roots of the secular equation $L_n(x) = 0$ are all real.‡ Weber's proof as we shall see, is complicated by his belief that it is necessary to show that

$$L_n'(x) = -\sum_i \frac{\partial L_n}{\partial a_{ii}}, \qquad (i = 1, 2, \cdots, n).$$

* Presented to the Society, December 29, 1926.

† See T. J. Bromwich, *Quadratic Forms*, Cambridge Tracts, No. 3 (1906), p. 69.

‡ H. Weber, *Algebra*, vol. 1, 1898, pp. 307–310.

2. *Proof of the Theorem.* We turn now to the proof of the above Theorem I. Consider the sequence

$$(2)\qquad H_n, H_{n-1}, H_{n-2}, \cdots, H_1, H_0 = 1,$$

where H_k is the determinant of degree k in x obtained by deleting the last $n-k$ rows and columns of (1). For the moment we suppose that no two of the H's vanish for the same x. They are connected by the relations

$$(3)\qquad \begin{aligned} H_n H_{n-2} &= H_{n-1}\phi_{n-1} - \psi_{n-1}^2, \\ H_{n-1} H_{n-3} &= H_{n-2}\phi_{n-2} - \psi_{n-2}^2, \\ &\cdots\cdots\cdots \\ H_2 H_0 &= H_1\phi_1 - \psi_1^2, \end{aligned}$$

where ϕ, ψ are polynomials in x.

Merely for completeness let us show how these relations are obtained, the first for example. Let A_{ij} be the minor of a_{ij} in (1); set $\nu = n-1$. Then

$$(4)\qquad B = \begin{vmatrix} A_{\nu\nu} & A_{\nu n} \\ A_{n\nu} & A_{nn} \end{vmatrix} = H_{n-1} A_{\nu\nu} - A_{n\nu}^2 .$$

But $H_n \cdot B = H_{n-2} \cdot H_n^2$. Hence, if $H_n \neq 0$, $B = H_{n-2} \cdot H_n$. This with (4) gives

$$H_n H_{n-2} = H_{n-1}\phi_{n-1} - \psi_{n-1}^2.$$

This relation holds also if $H_n = 0$, as continuity considerations show.*

The equations (3) show that when $H_k = 0$, H_{k+1}, H_{k-1} have opposite signs.

We now consider the signs of the sequence (2). Suppose $H_n = 0$ for $x = a$. Then in a sufficiently small interval δ about $x = a$, H_n changes its sign, while none of the other terms in (2) do. Thus as x passes through δ the sequence (2) gains or loses one variation of sign. On the other hand when x passes through a root of $H_{n-1}, H_{n-2}, \cdots$ no variation is gained or lost as (3) show. For $x = +\infty$ there are r

* See Weber, loc. cit., p. 113; or Kowalewski, *Determinantentheorie*, p. 83.

variations of sign in (2); for $x = -\infty$ there are s, thus $H_n(x) = 0$ has at least σ real roots.

We now consider the general case that the sequence (2) has common roots. With Weber we may dispose of this case as follows. Suppose e.g. that H_k, H_{k-1} have common roots. We vary the terms a_{ij} of H_k not in H_{k-1} by small amounts numerically less than some η, so that H_k, H_{k-1} do not have common roots.

In this way we may replace (2) by another sequence

$$K_n, K_{n-1}, K_{n-2}, \cdots, K_1, K_0 = 1 \tag{5}$$

no two of which have a common root. The roots of $K_n = 0$ differ from those of $H_n = 0$ by an amount as small as we please, for sufficiently small η, moreover the signs of corresponding elements of the sequences (2), (5) are the same for an x for which no element of (2) vanishes. As Theorem I holds for (5), it must hold for (2).

THEOREM II. *The roots of the secular equations are all real.*

For in this equation $s = 0$; hence $\sigma = n$.

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A GENERALIZED TWO-DIMENSIONAL POTENTIAL PROBLEM

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It may be shown that the solution of the problem of electromagnetic wave propagation along a system of straight parallel conductors can be reduced to the solution* of two subsidiary problems: (1) a well known problem in two-dimensional potential theory; and (2) a generalization of the two-dimensional potential problem which is believed to be novel. The generalized problem is believed to possess suffi-

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* Subject to certain restrictions to be discussed in a forthcoming paper.